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Kinematic and Dynamic Synthesis of Geared Robotic Mechanisms

This paper describes a methodology for the design of geared robotic mechanisms. It is shown that certain gear-coupled manipulators can be designed to possess kinematic isotropy property at a given end-effector position. For these gear-coupled manipulators, the train values can be treated as a product of two-stage gear reductions. The second-stage reduction can be uniquely determined from the kinematic isotropic condition, while the first-stage reduction can be determined from dynamic consideration. This approach, through proper choice of gear ratios, can provide these gear-coupled manipulators with desired kinematic and dynamic characteristics.

Introduction

Various performance measures have been proposed for the evaluation of kinematic and/or dynamic performance of a manipulator. Most of the kinematic performance measures, such as the velocity ellipsoid (Asada and Cro Granlto, 1985; Dubey and Luh, 1986), the generalized velocity ratio (Asada and Cro Granlto, 1985; Dubey and Luh, 1986), the manipulability measure (Yoshikawa, 1985a), and the condition number (Gosselin and Angeles, 1988), are based on the relation between velocity vectors in the joint-space and end-effector-space of an open-loop manipulator. As for the dynamic performance measure, Yoshikawa (1985b) proposed a dynamic manipulability index which defines the relation between joint torque and the end-effector acceleration. Since these performance measures are based on the transformation between the joint-space and end-effector-space, they can be used for the evaluation or design of direct-drive manipulators. However, they are not very helpful in evaluating manipulators which use gear trains or other means for power transmission.

For geared robotic mechanisms, the transformation between the actuator-space and joint-space must also be taken into consideration. That is, the transformation has to be extended from between "end-effector-space and joint-space" to "end-effector-space and actuator-space." The structure matrix, defined by Chang and Tsai (1989), transforms the velocity vector from joint-space to actuator-space while the Jacobian matrix transforms the velocity vector from joint-space to end-effector-space. Together, they give the overall transformation from actuator-space to end-effector-space.

In what follows, the definitions of various performance measures will be extended from direct-drive manipulators to nondirect-drive manipulators and, in particular, gear-coupled manipulators. The necessary condition for kinematically isotropic transformation will be derived. The performance evaluation problem will be extended to a design optimization problem. Finally, equations for train values determination will be derived by taking both kinematics and dynamics into consideration.

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Kinematic Characteristics

Generalized Velocity Ratio. The velocity ratio and the mechanical advantage are the two most commonly used criteria for evaluating the performance of a single-input and single-output mechanism such as the four bar linkage. The velocity ratio is the ratio of output velocity to input velocity and the mechanical advantage is the ratio of output torque to input torque at the instant of interest. For n-DOF (degree-of-freedom) mechanisms, the concept of velocity ratio and mechanical advantage has been extended to that of generalized velocity ratio and generalized mechanical advantage. Specifically, the magnitude of input velocity vector is compared to that of the output velocity vector.

Figure 1 shows a geared robotic mechanism in conceptual form, where the inputs to the mechanism are the actuators and the output is the end-effector. Let $\mathbf{\Phi}$, $\mathbf{\Theta}$, and $\mathbf{X}$ be the displacement vectors associated with the actuators, joints, and the end-effector. Let $\dot{\mathbf{\Phi}}$, $\dot{\mathbf{\Theta}}$, and $\dot{\mathbf{X}}$ be the time derivatives of $\mathbf{\Phi}$, $\mathbf{\Theta}$, and $\mathbf{X}$. And let ξ, τ, and $\mathbf{F}$ be the generalized force vectors in the actuator-space, joint-space and end-effector-space, respectively. Then, the joint and output velocity vectors are related by the Jacobian matrix, J, as

$$\dot{\mathbf{X}} = J\dot{\mathbf{\Theta}}, \tag{1}$$

and the joint torque and output force vectors are related by

$$\tau = J^T\mathbf{F} \tag{2}$$

where $(\ )^T$ denotes the transpose of ( ).

The actuator and joint velocity vectors are related by the structure matrix, A, as

$$\dot{\mathbf{\Phi}} = A^T\dot{\mathbf{\Theta}}, \tag{3}$$

and the joint and actuator torque vectors are related by

$$\tau = A\xi, \tag{4}$$

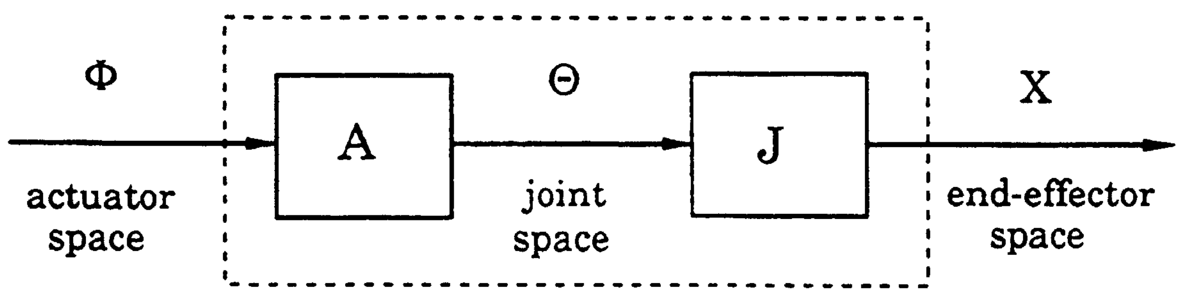

Fig. 1 Conceptual block diagram of a geared robotic mechanism

where the elements of A are functions of gear ratios in a mechanism. The ith row of the structure matrix A describes how the resultant torque about joint "i" is effected by the input actuators and, on the other hand, the jth column of matrix A describes how the torque of an input actuator "j" is transmitted to various joints of a mechanism. We note that the velocity vector, $\dot{\mathbf{X}}$, in Eq. (1) contains both linear and angular velocities of a point in the end-effector. Similarly, the force vector, $\mathbf{F}$, in Eq. (2) contains both forces and couples acting on a point in the end-effector.

In general, the elements in a velocity vector may have different units. Hence, it is necessary to define a weighted norm for the magnitude of a velocity vector. In this paper, the following quadratic forms are defined for the square of the norms:

$$|\dot{\mathbf{X}}|^2 \equiv \dot{\mathbf{X}}^T W_x \dot{\mathbf{X}} \tag{5}$$

and

$$|\dot{\mathbf{\Phi}}|^2 \equiv \dot{\mathbf{\Phi}}^T W_\phi \dot{\mathbf{\Phi}} \tag{6}$$

where W_x and W_ϕ are diagonal, positive, definite, weighting matrices.

As an extension, the square of the generalized velocity ratio K_v is defined as the ratio of the two quadratics:

$$K_v^2 \equiv \frac{|\dot{\mathbf{X}}|^2}{|\dot{\mathbf{\Phi}}|^2} \tag{7}$$

Substituting Eqs. (1) and (3) into (5) and (6), we obtain

$$|\dot{\mathbf{X}}|^2 = \dot{\mathbf{\Theta}}^T J^T W_x J \dot{\mathbf{\Theta}} = \dot{\mathbf{\Phi}}^T A^{-1} J^T W_x J A^{-T} \dot{\mathbf{\Phi}} \tag{8}$$

and

$$|\dot{\mathbf{\Phi}}|^2 = \dot{\mathbf{\Theta}}^T A W_\phi A^T \dot{\mathbf{\Theta}} \tag{9}$$

where $(\ )^{-1}$ denotes the inverse of $(\ )$, and $(\ )^{-T}$ the inverse of $(\ )^T$.

From Eqs. (6), (7), (8), and (9), we obtain

$$K_v^2 = \frac{\dot{\mathbf{\Phi}}^T A^{-1} J^T W_x J A^{-T} \dot{\mathbf{\Phi}}}{\dot{\mathbf{\Phi}}^T W_\phi \dot{\mathbf{\Phi}}} \tag{10}$$

or

$$K_v^2 = \frac{\dot{\mathbf{\Theta}}^T J^T W_x J \dot{\mathbf{\Theta}}}{\dot{\mathbf{\Theta}}^T A W_\phi A^T \dot{\mathbf{\Theta}}} \tag{11}$$

Equations (10) and (11) are known as Rayleigh's quotient. The value of K_v depends on the position as well as direction of motion of the end-effector. The extreme values of K_v are the square root of the eigenvalues of the following eigenvalue problem (Strang, 1980):

$$(W_\phi^{-1} A^{-1} J^T W_x J A^{-T}) \dot{\mathbf{\Phi}} = \lambda \dot{\mathbf{\Phi}} \tag{12}$$

or

$$(J^T W_x J) \dot{\mathbf{\Theta}} = \lambda (A W_\phi A^T) \dot{\mathbf{\Theta}} \tag{13}$$

Equations (12) and (13) have the same eigenvalues, λ's, and their eigenvectors are related by Eq. (3). Hence, the eigenvalues of Eq. (12) or (13) completely characterize the kinematic performance of a manipulator at a given end-effector position.

Isotropic Condition. Equations (12) or (13) can also be used for design optimization. Suppose the kinematic structure of a manipulator has already been selected and the problem is to define the gear ratios such that the generalized velocity ratio is less directional sensitive. This problem can be solved by minimizing the difference between the maximum and minimum eigenvalues of Eq. (12) or (13). Equation (12) contains both Jacobian and structure matrices on the left-hand side of the equation, while (13) contains the Jacobian matrix on the left-hand side and the structure matrix on the right-hand side. The separation of Jacobian matrix from structure matrix makes it more convenient to use Eq. (13) for the purpose of design

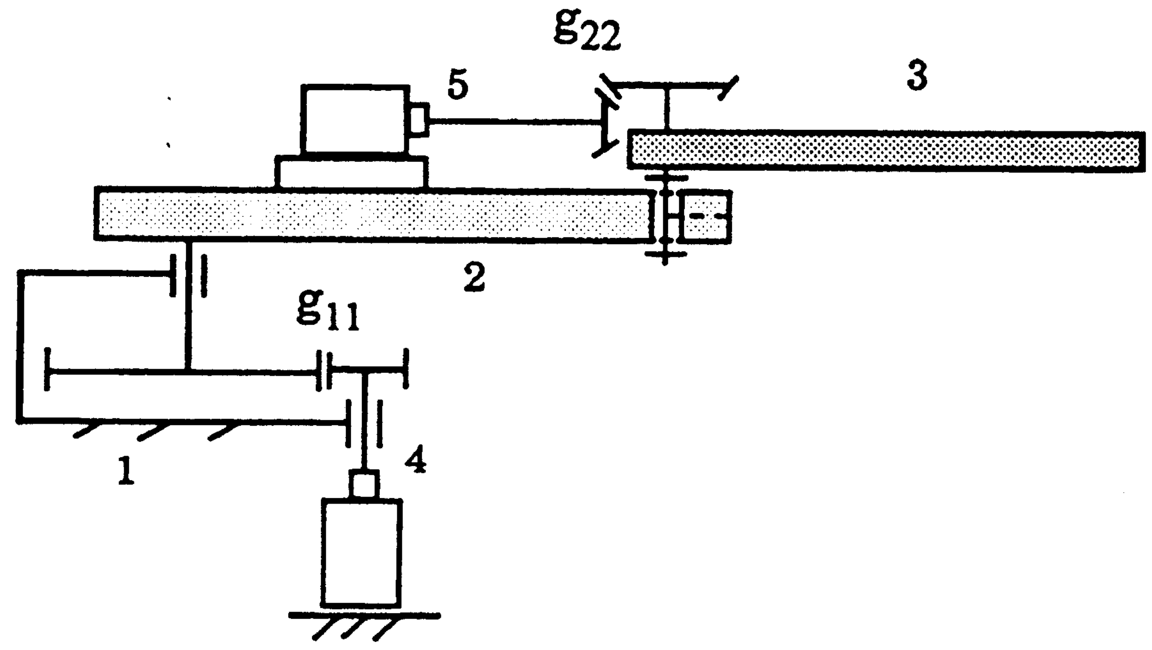

Fig. 2 Schematic diagram of a two-DOF planar individual joint-drive manipulator

optimization. For Eq. (13) to have nontrivial solutions, the following condition must be satisfied:

$$det(P - \lambda Q) = 0 \tag{14}$$

where $P = J^T W_x J$ and $Q = A W_\phi A^T$.

Since both P and Q are positive definite matrices, the eigenvalues, λ's, are all positive real numbers. It has been shown that for λ to be an r-fold root, all the principal minors of $(P - \lambda Q)$ starting from order n to order $n - r + 1$ must vanish (Jeffreys, 1956; Goldstein, 1981). If λ is an n-fold root for an n-DOF geared robotic mechanism, then the mechanism is said to be kinematically isotropic at the given end-effector position. Under this condition the generalized velocity ratio, $K_v = \sqrt{\lambda}$, is independent of the direction of motion. For λ to be an n-fold root, the following proportional condition must be satisfied

$$(J^T W_x J)_{i,j} = \lambda (A W_\phi A^T)_{i,j} \tag{15}$$

where $(\ )_{i,j}$ denotes the (i, j) element of the matrix enclosed in the parenthesis.

Individual Joint-Drive Manipulators. If every moving link in a manipulator is driven by an actuator mounted on its preceding link through a gear-reduction unit such as the one shown in Fig. 2, then the joint motions are independent of each other. We call this type of manipulators *individual joint-drive manipulators*. The structure matrix for an individual joint-drive manipulator has the following form:

$$A = \begin{bmatrix} g_{11} & & & 0 \\ & g_{22} & & \\ & & \ddots & \\ 0 & & & g_{nn} \end{bmatrix} \tag{16}$$

where g_{ii}, is the gear reduction for the ith actuator. Hence,

$$A W_\phi A^T = \begin{bmatrix} w_1 g_{11}^2 & & & 0 \\ & w_2 g_{22}^2 & & \\ & & \ddots & \\ 0 & & & w_n g_{nn}^2 \end{bmatrix} \tag{17}$$

where w_i is the (i, i) element of W_ϕ.

At a given end-effector position, the product of Jacobian matrix can be written as

$$J^T W_x J = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \cdots & \epsilon_{1n} \\ \epsilon_{12} & \epsilon_{22} & \cdots & \epsilon_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon_{1n} & \epsilon_{2n} & \cdots & \epsilon_{nn} \end{bmatrix} \tag{18}$$

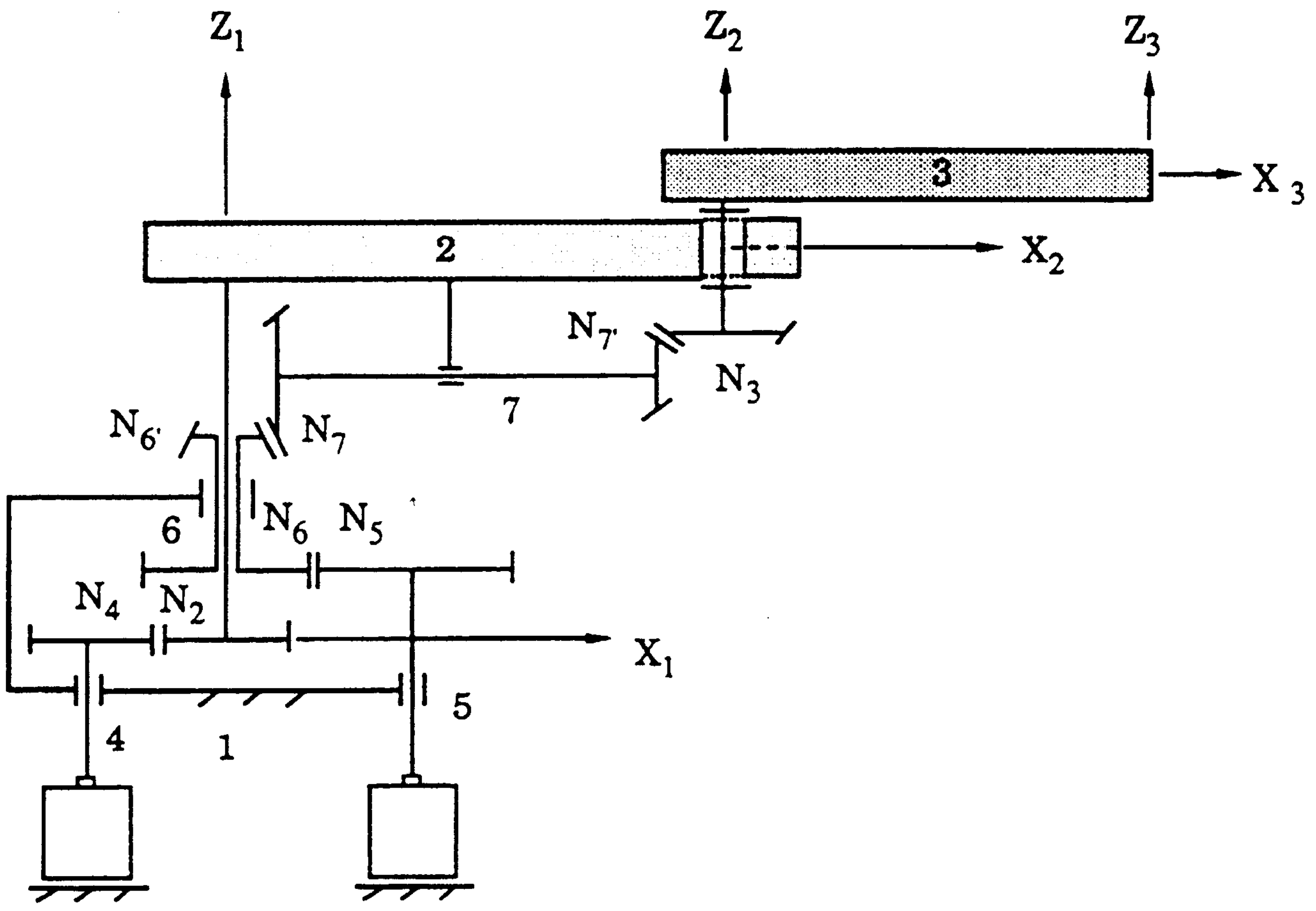

Fig. 3 Schematic diagram of a two-DOF planar gear-coupled manipulator

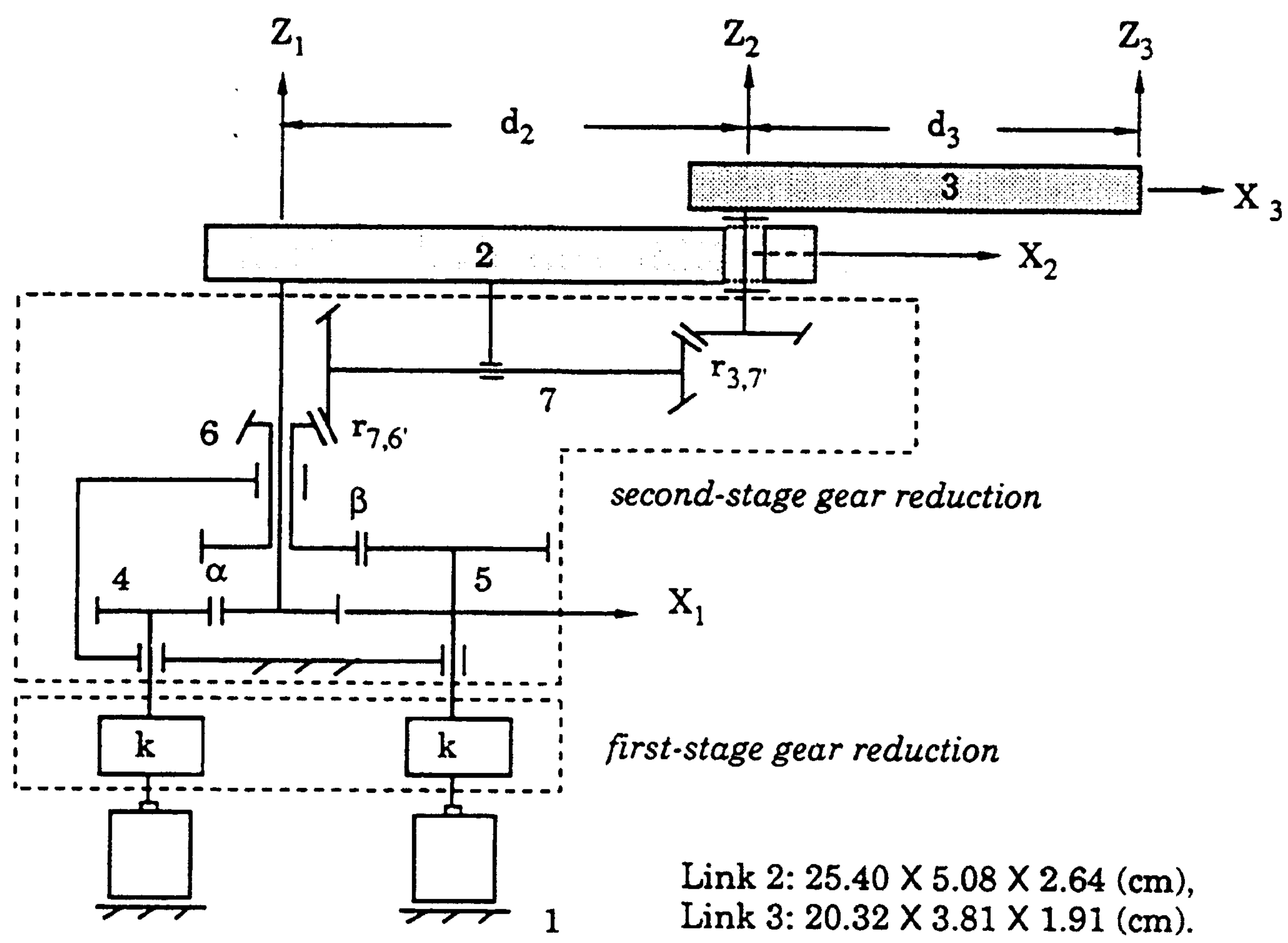

Fig. 4 Illustration of two-stage gear reductions of the manipulator shown in Fig. 3

Substituting Eqs. (17) and (18) into (15), yields

$$\epsilon_{ij} = \lambda \begin{cases} w_i g_{ii}^2, & i=j, \\ 0, & i \neq j. \end{cases} \tag{19}$$

It is obvious that Eq. (19) cannot be satisfied by any choice of g_{ii}, unless $\epsilon_{ij} = 0$ for all i not equal to j, which requires certain special link and joint parameters. This leads to the following theorem.

Theorem 1. Individual joint-drive manipulators cannot possess a kinematically isotropic property unless the product of Jacobian matrix, $J^T W_x J$, is a diagonal matrix at the position of interest.

Gear-Coupled Manipulators. If some of the links in a manipulator are driven by actuators mounted on links other than their preceding links through the use of gear trains, then the joint motions are coupled. We call this type of mechanisms *gear-coupled manipulators.*

The structure matrix for gear-coupled manipulator is nolonger diagonal (Chang and Tsai, 1989). For an n-DOF gear-coupled manipulator, Eq. (15) yields $n(n+1)/2$ nonlinear equations. However, the number of unknowns contained in Eq. (15) depends on the arrangement of transmission lines, i.e., the number of nonzero elements in the structure matrix. It is essential that the number of unknowns is not less than the number of equations. If the number of unknowns is less than the number of equations, then special linkage geometry is required to yield an isotropic condition. If the number of unknowns is greater than the number of equations, then there exist some free choices among the nonzero elements in the structure matrix. This leads to our second theorem.

Theorem 2. Gear-coupled manipulators can be designed to possess an isotropic property at a given end-effector position if and only if the number of nonzero element in the structure matrix is equal to or greater than $n(n+1)/2$.

Example. Figure 3 shows a two-DOF planar manipulator with both actuators mounted on the ground. There are two transmission lines. The first transmits an actuator torque through the (4, 2) gear pair. The second transmits another actuator torque through the (5, 6), (6, 7), and (7, 3) gear pairs. The structure matrix is given by

$$A = \begin{bmatrix} g_{11} & g_{12} \\ 0 & g_{12} r_{3,7'} r_{7,6'} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{bmatrix} \tag{20}$$

where $g_{11} = N_2/N_4$, $g_{12} = N_6/N_5$, $g_{22} = g_{12}\, r_{7,6'}\, r_{3,7'}$, and where $r_{7,6'} = N_7/N_{6'}$, $r_{3,7'} = N_3/N_{7'}$, and N_i denotes the number of tooth on gear i.

Assuming at a given end-effector position, the product of Jacobian matrix takes the following form:

$$J^T W_x J = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{12} & \epsilon_{22} \end{bmatrix} \tag{21}$$

and W_ϕ is an identity matrix, then it follows from Eqs. (15), (20), and (21) that

$$K_v^2 (g_{11}^2 + g_{12}^2) = \epsilon_{11} \tag{22a}$$

$$K_v^2 g_{12} g_{22} = \epsilon_{12} \tag{22b}$$

$$K_v^2 g_{22}^2 = \epsilon_{22} \tag{22c}$$

Solving Eqs. (22*a*)-(22*c*), we obtain

$$|g_{11}| = \frac{\sqrt{(\epsilon_{11}\epsilon_{22} - \epsilon_{12}^2)/\epsilon_{22}}}{K_v} \tag{23a}$$

$$|g_{12}| = \epsilon_{12}/(\sqrt{\epsilon_{22}}\, K_v) \tag{23b}$$

$$|g_{22}| = \sqrt{\epsilon_{22}}/K_v \tag{23c}$$

and where the signs of g_{11}, g_{12}, and g_{22} in A can have one of the following combinations:

$$A = \begin{bmatrix} + & + \\ 0 & + \end{bmatrix}, \text{ or } \begin{bmatrix} + & - \\ 0 & - \end{bmatrix}, \text{ or } \begin{bmatrix} - & + \\ 0 & + \end{bmatrix}, \text{ or } \begin{bmatrix} - & - \\ 0 & - \end{bmatrix}$$

Hence, a sign change along any transmission line does not change the isotropic condition. Alternatively, Eqs. (23*a*-23*c*) can be written in the following form:

$$g_{11} = \alpha k \tag{24a}$$

$$g_{12} = \beta k \tag{24b}$$

$$r_{3,7'}\, r_{7,6'} = g_{22}/g_{12} = \epsilon_{22}/\epsilon_{12} \tag{24c}$$

where

$$k = \frac{\sqrt{(\epsilon_{11}\epsilon_{22} - \epsilon_{12}^2)/\epsilon_{22}}}{\alpha K_v} = \frac{\sqrt{det(J^T W_x J)}}{\alpha K_v \sqrt{\epsilon_{22}}} \tag{25a}$$

$$\beta = \alpha\epsilon_{12}/\sqrt{(\epsilon_{11}\epsilon_{22} - \epsilon_{12}^2)} = \alpha\epsilon_{12}/\sqrt{det(J^T W_x J)} \tag{25b}$$

We note that α can be chosen arbitrarily. But, once α is chosen, β is determined by Eq. (25*b*). It follows from Eqs. (24) and (25) that k, which is inversely proportional to the generalized velocity ratio K_v, can be considered as a scaling factor and the train value for each transmission line can be thought of as a product of two-stage gear reductions as shown in Fig. 4. The first-stage gear reduction, k, which is common to all transmission lines, provides the desired overall reduction while the second-stage gear reduction provides the necessary condition for an isotropic transformation.

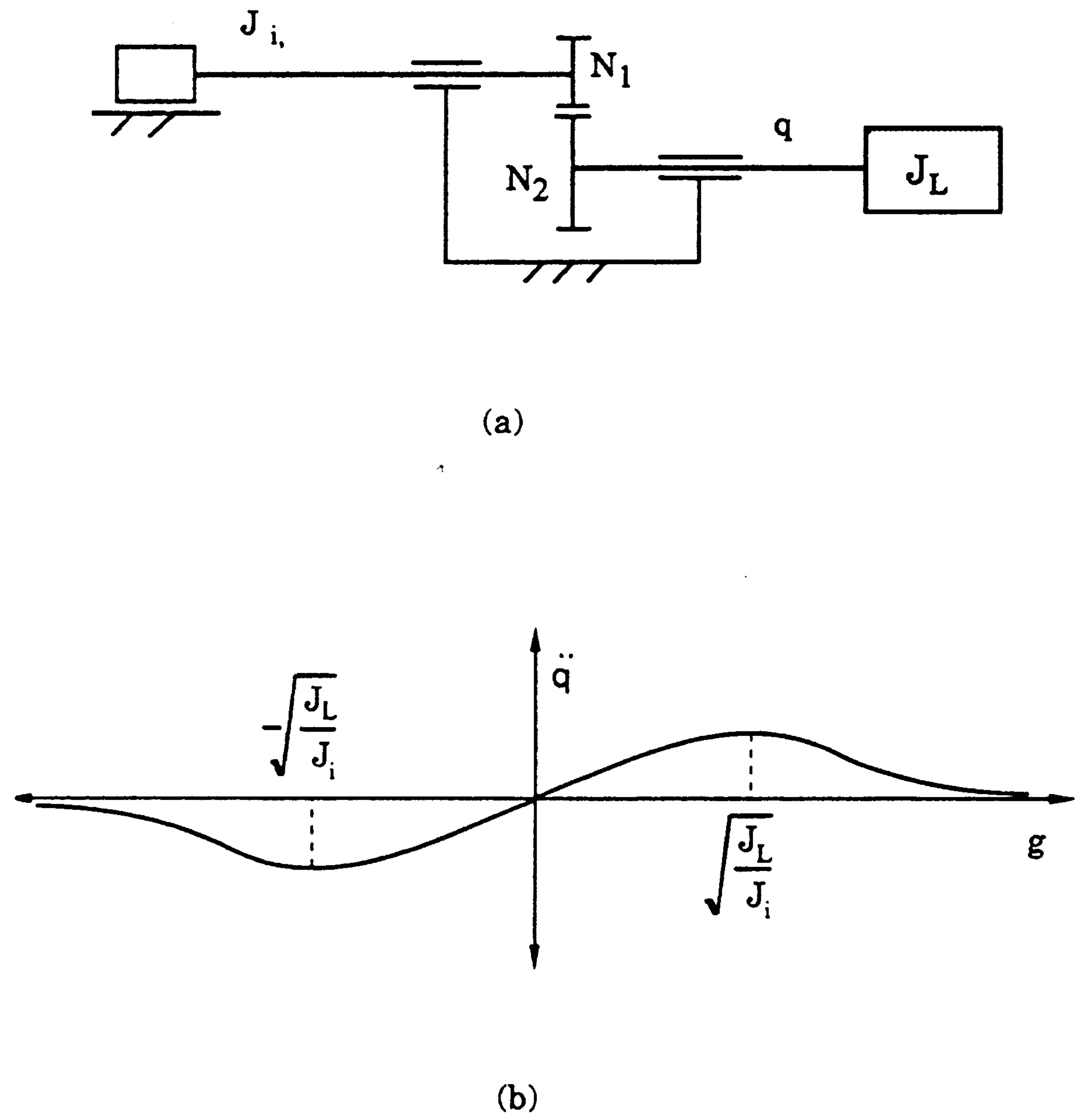

Fig. 5(*a*) A one-DOF gear mechanism, (*b*) Variation of output acceleration vs. gear ratio

For the manipulator shown in Fig. 4, it can be shown that the Jacobian matrix is given by

$$J=\begin{bmatrix} -d_3S_{12}-d_2S_1 & -d_3S_{12} \\ d_3C_{12}+d_2C_1 & d_3C_{12} \end{bmatrix} \tag{26}$$

where $d_2 = 22.86$ cm, $d_3 = 17.78$ cm are the lengths of link 2 and link 3, respectively, and where S_i, C_i, S_{12}, and C_{12} denote $\sin(\theta_i)$, $\cos(\theta_i)$, $\sin(\theta_1 + \theta_2)$, and $\cos(\theta_1 + \theta_2)$, respectively. Hence, with the end-effector positioned at $[X_1, Y_1] = [22.86, 0]$, we have

$$J=\begin{bmatrix} 0 & 16.38 \\ 22.86 & 6.91 \end{bmatrix} \tag{27}$$

Assuming W_x and W_ϕ are both identity matrices, we have

$$J^TW_xJ=\begin{bmatrix} 522.58 & 157.96 \\ 157.96 & 316.05 \end{bmatrix} \tag{28}$$

Substituting Eq. (28) into (24) and (25), we obtain

$$k=21.063/(\alpha K_v) \tag{29a}$$

$$\beta=0.422\ \alpha \tag{29b}$$

$$r_{3,7'}\ r_{7,6'}=2 \tag{29c}$$

or

$$A=\frac{21.063}{K_v}\begin{bmatrix} \alpha & 0.422\ \alpha \\ 0 & 0.844\ \alpha \end{bmatrix} \tag{30}$$

For example, we can choose $\alpha = 1$ and $r_{3,7'} = 1$, then $\beta = 0.422$ and $r_{7,6'} = 2$. Hence, a designer can finalize the second-stage gear reduction without concerning the generalized velocity ratio, K_v.

Dynamic Characteristics

In the previous section, we have shown that infinite many sets of gear ratios can be used to produce a kinematically isotropic condition for those gear-coupled manipulators which satisfy theorem 2. This leaves additional room for dynamic optimization.

Principle of Inertia Match and Acceleration Capacity. For a one-DOF geared mechanism as shown in Fig. 5(*a*), the equation of motion can be written as

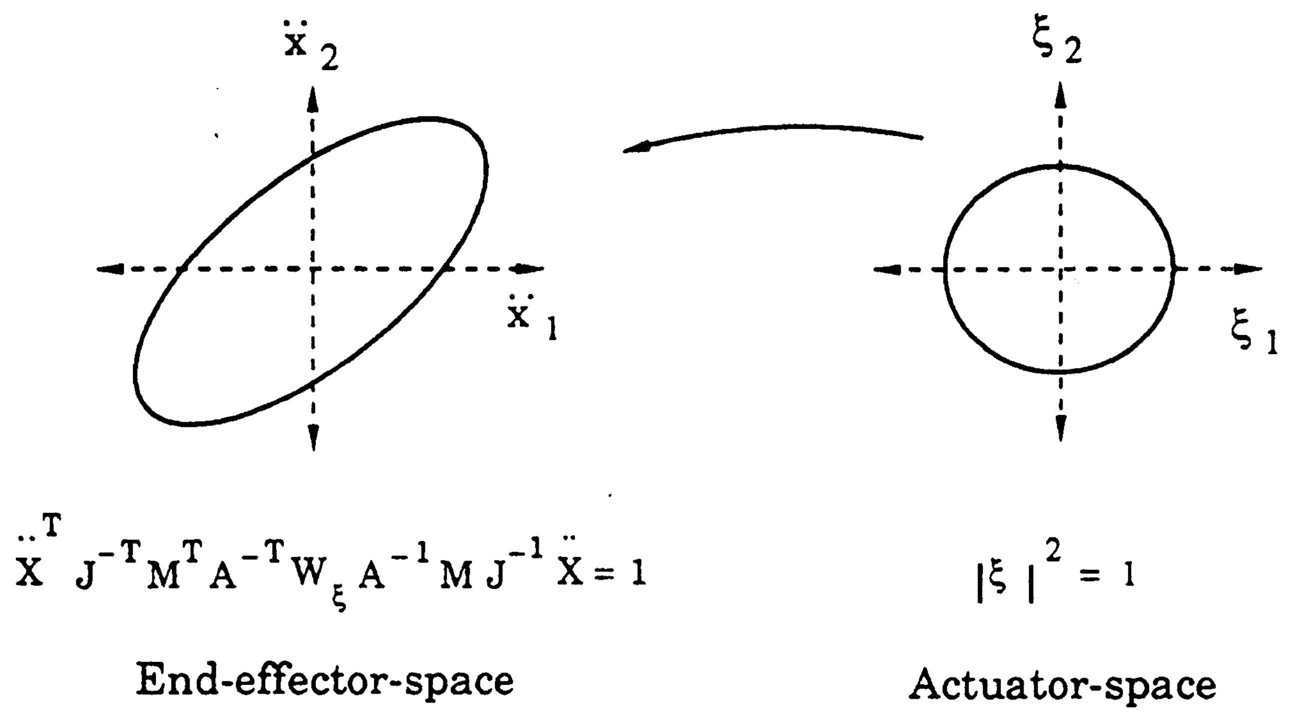

Fig. 6 Transformation between actuator-space and end-effector-space

$$(J_L+g^2J_i)\ddot{q}=g\xi_i \tag{31}$$

where J_L denotes the load inertia, J_i the rotor inertia of input actuator, ξ_i the input torque, q the angular displacement of the output shaft, and $g = N_2/N_1$ the gear ratio.

Assume that J_i and J_L remain constant regardless of the change in gear ratio and assume that there is no power loss in the gear mesh. Figure 5(*b*) shows the relation between the output shaft acceleration, $\ddot{q}$, and the gear ratio, g. It is clear that, given ξ_i, J_L, and J_i, there exists an optimum gear ratio which yields a maximum output acceleration. At the optimum design, the output acceleration and the gear ratio are given by

$$\ddot{q})_{max}=\frac{\xi_i}{2\sqrt{J_LJ_i}} \tag{32a}$$

$$g^2_{opt}=\frac{J_L}{J_i} \tag{32b}$$

Equations (32*a*) and (32*b*) can be simply stated as follows. At the optimum design, the gear ratio is chosen such that the reflected input inertia is "matched" with the output inertia. This is known as the "principle of inertia match" (Stockdale, 1968).

For an *n*-DOF geared robotic mechanism, the equations of motion can be written in the joint-space as

$$M\ddot{\boldsymbol{\theta}}+\dot{\boldsymbol{\theta}}^TC\dot{\boldsymbol{\theta}}+\mathbf{G}=A\xi \tag{33}$$

where M is an n by n inertia matrix, $\dot{\boldsymbol{\theta}}^TC\dot{\boldsymbol{\theta}}$ is the generalized inertia force contributed by the coriolis and centrifugal effects, and G is the generalized active force contributed by gravitational effect and/or external loads (Chen et al., 1990).

In what follows, we shall neglect the coriolis and centrifugal effects, and we shall also assume that there are no gravitational forces and external loads. Then, Eq. (33) can be simplified as

$$M\ddot{\boldsymbol{\theta}}=A\xi \tag{34}$$

Differentiating Eq. (1) and neglecting the coriolis and centrifugal accelerations, we obtain

$$\ddot{\mathbf{X}}=J\ddot{\boldsymbol{\theta}} \tag{35}$$

Eliminating $\ddot{\boldsymbol{\theta}}$ from Eqs. (34) and (35), yields

$$A^{-1}MJ^{-1}\ddot{\mathbf{X}}=\xi \tag{36}$$

Equation (36) provides a torque transformation from the end-effector-space to the actuator-space. In this paper, the following quadratic forms are defined for the square of the norm of the input torque and end-effector acceleration.

$$|\xi|^2\equiv\xi^TW_\xi\xi \tag{37a}$$

$$|\ddot{\mathbf{X}}|^2\equiv\ddot{\mathbf{X}}^TW_x\ddot{\mathbf{X}} \tag{37b}$$

where W_ξ is a diagonal, positive definite, weighting matrix. In general, W_ξ is chosen as the inverse of W_ϕ, i.e., $W_\xi W_\phi = I$.

Substituting Eq. (36) into (37*a*), we obtain

$$|\xi|^2=\ddot{\mathbf{X}}^TJ^{-T}M^TA^{-T}W_\xi A^{-1}MJ^{-1}\ddot{\mathbf{X}} \tag{38}$$

Hence, at a given posture, $|\xi|^2 = 1$ yields an acceleration ellipsoid in the end-effector-space as shown in Fig. 6. The acceleration capacity, A.C., is defined to be proportional to the volume of the ellipsoid, i.e.,

$$\text{A.C.} \equiv 1/\left(\prod_{i=1}^{n} \sqrt{\mu_i}\right) \tag{39}$$

where μ_i, $i = 1, 2, 3, \ldots, n$, are the eigenvalues of the following eigenvalue problem:

$$(W_x^{-1}J^{-T}M^TA^{-T}W_\xi A^{-1}MJ^{-1})\ddot{\mathbf{X}} = \mu\ddot{\mathbf{X}} \tag{40}$$

It can be shown that (Strang, 1980) the acceleration capacity, A.C., is equal to one over the square root of determinant of the matrix, i.e.,

$$\text{A.C.} = 1/[det(W_x^{-1}J^{-T}M^TA^{-T}W_\xi A^{-1}MJ^{-1})]^{1/2}$$
$$= \frac{[det(J^TW_xJ)det(AW_\phi A^T)]^{1/2}}{det(M)} \tag{41}$$

Substituting Eq. (15) into (41), we obtain

$$\text{A.C.} = \frac{det(J^TW_xJ)}{K_v^n det(M)} \tag{42}$$

The acceleration capacity, A.C., can be used as an index to indicate the ability of a manipulator to respond to a given set of input torques. The larger the acceleration capacity, the more responsive the system is. At a given end-effector position the determinant of the product of Jacobian matrix, $det(J^TW_xJ)$, is a constant while the determinant of inertia matrix, $det(M)$, is a function of gear ratios. Hence, the unknown gear ratios can then be used to optimize the acceleration capacity.

Acceleration Capacity Optimization. The inertia matrix M in Eq. (33) can be divided into two parts, namely, the contribution of the major links in the equivalent open-loop chain, M_m, and that of the carried links, M_c (Chen et al., 1990).

$$M = M_m + M_c \tag{43}$$

Considering the manipulator shown in Fig. 4 as an example. Let J_i be the axial moment of inertia of gear i, $\mathbf{P}_2 = [p_{2x}, p_{2y}]^T$ the position vector of the combined mass center of link 2 and gear 7 expressed in the link 2 coordinate system, $\mathbf{P}_3 = [p_{3x}, p_{3y}]^T$ the position vector of link 3 expressed in the link 3 coordinate system, m_2 the combined mass of link 2 and gear 7, m_3 the mass of link 3, I_{2z} the combined moment of inertia of link 2 and gear 7 about the Z_2-axis, and I_{3z} the moment of inertia of link 3 about Z_3-axis. Then, it can be shown that

$$M_m = \begin{bmatrix} m_{m11} & m_{m12} \\ m_{m12} & m_{m22} \end{bmatrix} \tag{44}$$

where

$$m_{m11} = m_2 d_2'^2 + I_{2z} + m_3(d_3'^2 + 2d_2d_3'C_2 + d_2^2) + I_{3z} \tag{45a}$$
$$m_{m12} = m_3(d_3'^2 + d_2d_3'C_2) + I_{3z} \tag{45b}$$
$$m_{m22} = m_3 d_3'^2 + I_{3z} \tag{45c}$$

and

$$M_c = k^2\begin{bmatrix} \delta_1 & \delta_2 \\ \delta_2 & \delta_3 \end{bmatrix} + \begin{bmatrix} J_6 & J_6 r_{3,7',7,6'} \\ J_6 r_{3,7',7,6'} & J_6(r_{3,7',7,6'})^2 + J_7 r_{3,7'}^2 \end{bmatrix} \tag{46}$$

where

$$d_2' = (p_{2x} + d_2) \tag{47a}$$
$$d_3' = (p_{3x} + d_3) \tag{47b}$$
$$\delta_1 = J_4\alpha^2 + J_5\beta^2 \tag{47c}$$
$$\delta_2 = J_5\beta^2 r_{3,7',7,6'} \tag{47d}$$
$$\delta_3 = J_5(\beta r_{3,7',7,6'})^2 \tag{47e}$$

Note that $r_{7,6',3,7'} = 2$, and the contribution of axial moments of inertias of gears 6 and 7 to the overall inertia can be neglected due to the small values of J_6, J_7, and $r_{7,6',3,7'}$. However, J_4 and J_5 can have significant effect on the overall inertia due to the k^2 term in Eq. (46). Hence, the inertia matrix contributed by carried links can be approximated by

$$M_c = k^2\begin{bmatrix} \delta_1 & \delta_2 \\ \delta_2 & \delta_3 \end{bmatrix} \tag{48}$$

In what follows, we shall assume that adjusting a gear ratio does not have significant effect on the mass and moment of inertia of the gear pair. Substituting Eqs. (44) and (46) into Eq. (43), the determinant of inertia matrix can be written as

$$det(M) = det(M_m) + k^2\rho_1 + k^4\rho_2 \tag{49}$$

where

$$\rho_1 = m_{m11}\delta_3 - 2m_{m12}\delta_2 + m_{m22}\delta_1 \tag{50a}$$
$$\rho_2 = \delta_1\delta_3 - \delta_2^2 \tag{50b}$$

Substituting Eqs. (25a) and (49) into (42), we obtain

$$\text{A.C.} = \alpha^2\epsilon_{22}\left[\frac{det(M_m)}{k^2} + \rho_1 + k^2\rho_2\right]^{-1} \tag{51}$$

It follows from Eq. (51) that, for a given manipulator posture, the acceleration capacity is a function of the first-stage gear reduction, k. Taking the derivative of Eq. (51) with respect to k and equating the resulting equation to zero, we obtain

$$k^4 = det(M_m)/\rho_2 \tag{52}$$

Substituting Eqs. (47c–47e) into Eq. (50b) and then the resulting equation into Eq. (52), the first-stage gear reduction, k can be solved as

$$k = \left(\frac{det(M_m)\epsilon_{12}^2}{J_4J_5\alpha^2\beta^2\epsilon_{22}^2}\right)^{1/4} \tag{53}$$

Taking second derivative of Eq. (51) with respect to k and substituting Eq. (52) into it, we obtain

$$\frac{\partial}{\partial k}\left(\frac{\partial \text{A.C.}}{\partial k}\right)_{k^4 = \frac{det(M_m)}{\rho_2}} = -\frac{8\alpha^2\epsilon_{22}\rho_2\sqrt{det(M_m)}}{[\rho_1 + 2\sqrt{\rho_2 det(M_m)}]^2} \tag{54}$$

Since ρ_2 and ϵ_{22} are positive, Eq. (54) will always be negative. Hence, Eq. (53) provides the optimum condition for maximum acceleration capacity of this planar two-DOF manipulator. At the optimum condition, the acceleration capacity is given by

$$\text{A.C.}_{opt} = \alpha^2\epsilon_{22}[\rho_1 + 2\sqrt{\rho_2 det(M_m)}]^{-1} \tag{55}$$

Assuming that $m_2 = 2.6$ kg, $m_3 = 1.156$ kg, $p_{2x} = -12.872$ cm, $p_{3x} = -10.201$ cm, $I_{2z} = 142.9$ kg-cm^2, $I_{3z} = 40.94$ kg-cm^2, $J_4 = J_5 = 8.79 * 10^{-2}$ kg-cm^2, for the manipulator shown in Fig. 4, then with the end-effector positioned at $[X_1, Y_1] = [22.86, 0]$, M_m is given by

$$M_m = \begin{bmatrix} 957.95 & 29.46 \\ 29.46 & 107.34 \end{bmatrix} (\text{kg-cm}^2) \tag{56}$$

Substituting Eqs. (56) and (28) into (53), and with $\alpha = 1$, $\beta = 0.422$, $r_{3,7'} = 1$, and $r_{7,6'} = 2$, we obtain $k = 65.59$ as the first-stage gear reduction. Hence, the generalized velocity ratio is given by $K_v = 0.32112$ cm.

Since the Jacobian matrix and the inertia matrix are position dependent, the isotropic property and maximum acceleration capacity obtained above are only local conditions. Usually, a reference position within the workspace is selected for design optimization. The performance of a manipulator will then vary from position to position. Hence, the reference position must be chosen carefully in order to achieve a good compromise between extreme positions. It seems that this can only be accomplished by an iterative process.

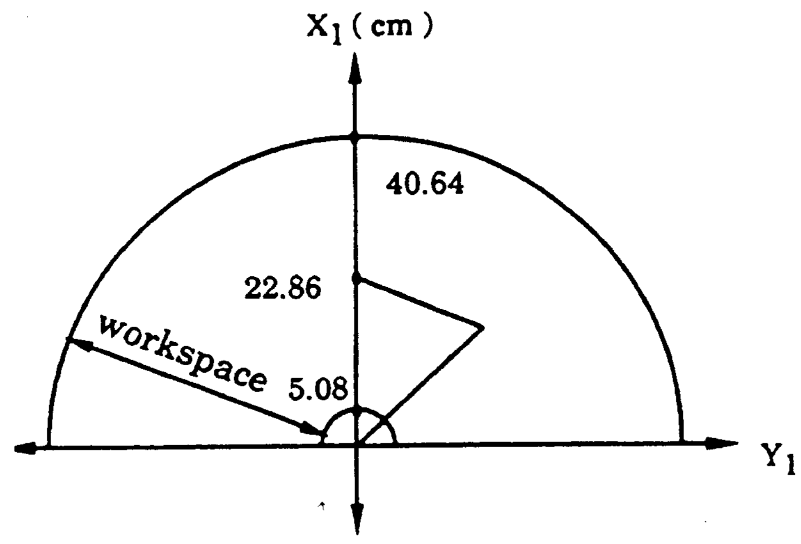

Fig. 7 Workspace of the manipulator shown in Fig. 4

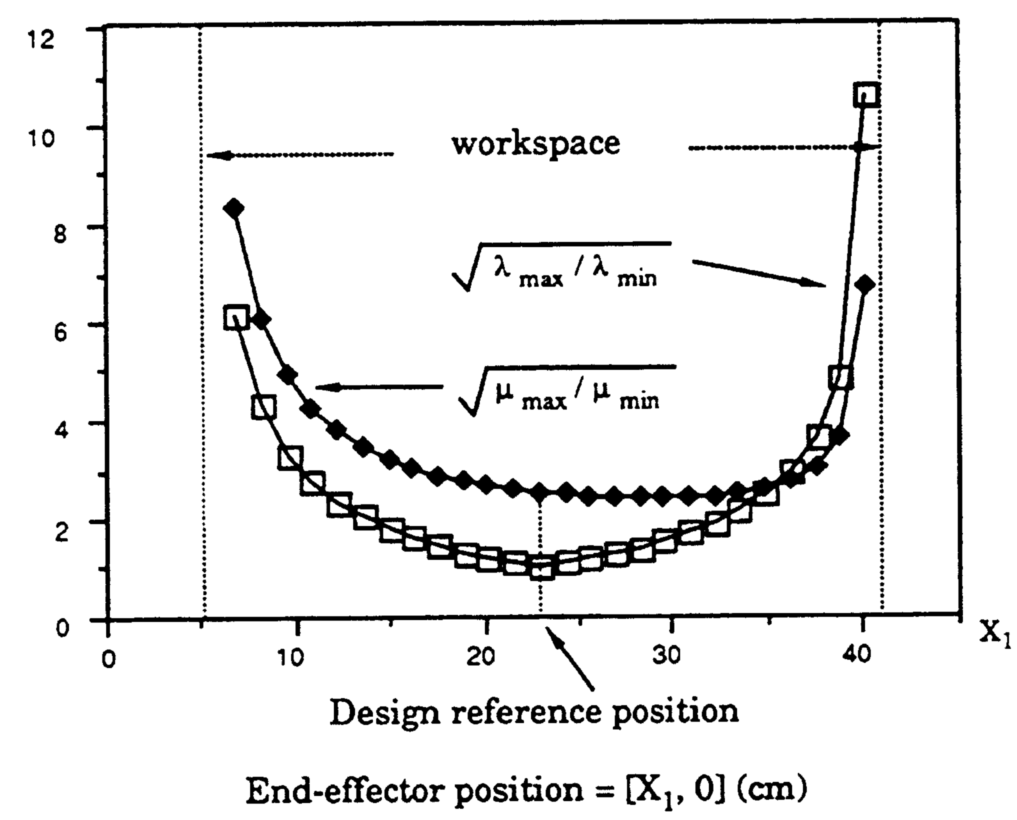

Fig. 8 Performances indices vs. end-effector position

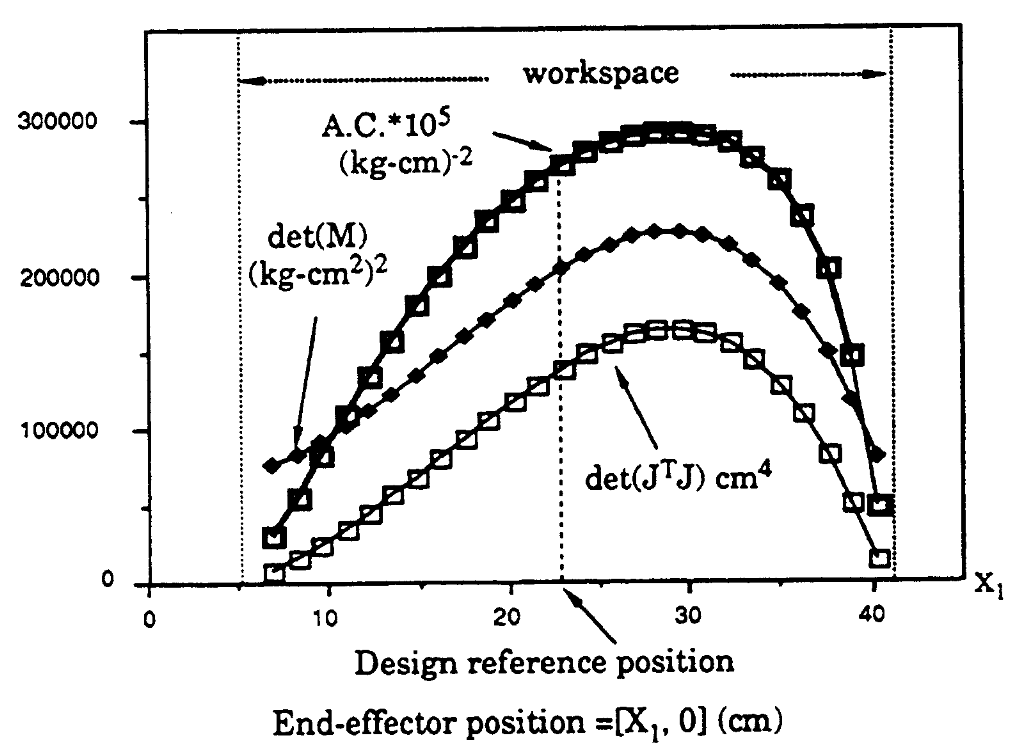

Fig. 9 Acceleration capacity (A.C.) vs. end-effector position

Figure 7 shows the workspace of the manipulator shown in Fig. 4. Since the Jacobian matrix and inertia matrix are symmetric about the first joint axis, it is only necessary to investigate the kinematic and dynamic performance along the X_1-axis. As a first approximation, the middle point of the workspace is chosen as the reference position for design optimization. Figure 8 shows the variation of the kinematic condition number ($\sqrt{\lambda_{max}/\lambda_{min}}$) and dynamic condition number ($\sqrt{\mu_{max}/\mu_{min}}$) as functions of the end-effector position. Since $X_1 = 22.86$ cm is chosen as the reference position, the global minimum kinematic condition number occurs at this reference position. However, the global minimum dynamic condition number does not occur at the reference position. Figure 9 shows the variation of the determinants of M and J^TJ, and the variation of the acceleration capacity, A.C., as functions of the end-effector position. As can be seen from Fig. 9, the global maximum acceleration capacity occurs at $X_1 = 29.53$ cm, instead of the reference position. This is due to the influence of the determinant of the product of the Jacobian matrix, J^TJ. Note that the maximum value of $det(J^TJ)$ occurs at $X_1 = 29.53$ cm coincidently.

Summary

We have derived a methodology for the determination of train values in geared robotic mechanisms. It is shown that certain gear-coupled manipulators can be designed to possess an isotropic condition at a given end-effector position. The train values of these gear-coupled manipulators can be thought of as a product of two-stage gear reductions. The second stage-gear reduction can be determined by the kinematic isotropic condition while the first-stage gear reduction can be determined by the maximum acceleration capacity condition. This approach can provide these gear-coupled manipulators with desired kinematic and dynamic characteristics.

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