Drive Train Configuration Arrangement for Gear Coupled Manipulators

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This paper describes a methodology for the drive train configuration arrangement of gear coupled manipulators. The approach is based on the idea that the kinematic structure of a gear coupled manipulator can be described by an equivalent open-loop chain (EOLC) and mechanical power transmission lines that drive the EOLC. Conditions to select proper drive train configuration such that manipulators can possess kinematic isotropic property will be presented. Procedures to select gear ratio based on optimum dynamic performance will also be extended from two-degree of freedom (DOF) systems to three-DOF systems. From the determined drive train configuration and gear ratios, location of actuators and details of the mechanical power transmission lines can be decided accordingly. A two-DOF manipulator and a three-DOF wrist are used as illustrative examples. *© 1997 John Wiley & Sons, Inc.*

ここでは、ギアが組み込まれたマニピュレータにおける、ドライブ・トレイン構成の編成方法を説明する。この方法は、ギアが組み込まれたマニピュレータの運動が、等価オープンループ・チエイン（EOLC）とそのEOLCを駆動する機械的動力の伝達線路によって説明できるという考え方に基づいている。そして、マニピュレータが理想的な力学特性を持つことができるように、適切なドライブ・トレイン構成を選択する条件について説明する。また、最適な力学的性能を得るギア比の選択を行う手順を拡張して、2 d. o. f.システムから3 d. o. f.のシステムに対して適用する。ドライブ・トレイン構成とギア比を設定すれば、アクチュエータの位置と機械的動力の伝達系の詳細が決定できる。なお、実例として、2 d. o. f.と3 d. o. f.のマニピュレータについて解説する。

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1. INTRODUCTION

The kinematic structure of a robot manipulator often takes the form of an open-loop configuration. An open-loop robot manipulator is mechanically simple and easy to construct. However, it does require the actuators to be located along the joint axes, which, in turn, degrades the dynamic performance of the system. For this reason, many robot manipulators are constructed in a partially closed-loop configuration to ease the actuator design and/or to reduce the inertia loads on the actuators. For the case of a gear coupled manipulator (GCM), gears are used to permit the actuators to be located as close to the base as possible and to transmit power to various joints of the open-loop chain. The open-loop chain of a GCM can be identified by removing all gears from it, and it is called the equivalent open-loop chain (EOLC).[1] Each link in the EOLC is referred to as a primary link; all the other links that are not rigidly attached to the primary links, including gears and actuator-rotors, are called the secondary links or carried links.[2] The arrangement of the secondary links, which describes where the input actuators are located and how the input torques are transmitted to various joints of the mechanism, forms the mechanical power transmission lines of the manipulator.

Through the mechanical power transmission line, which consists of spur or bevel gear trains, power is transmitted to the end-effector to which a payload can be attached. Chang and Tsai[3] showed that the structure matrix, which transforms the velocity vector from the joint-space to the actuator-space, describes the mechanical power transmission lines of a manipulator. They also showed that there are 30 admissible structure matrices for three degree-of-freedom (DOF) GCMs. Chen and Tsai[4] developed a methodology to determine the gear ratios based on kinematic isotropy and optimum acceleration capacity. However, their result is limited to two-DOF geared robotic mechanisms. Also, the kinematic isotropic condition can not be guaranteed for a given EOLC at a prescribed design reference point with an arbitrarily chosen structure matrix. Hence, only those structure matrices that are compatible with the given geometric property of the EOLC can be used as proper drive trains. In what follows, a methodology for the arrangement of drive train configuration of GCMs will be developed. It will be shown that GCMs can possess kinematic isotropic property and optimum dynamic performances through proper choice of drive train configuration and gear ratios. The condition for optimum acceleration capacity followed with kinematic isotropic property will be extended from two-DOF to three-DOF GCMs.

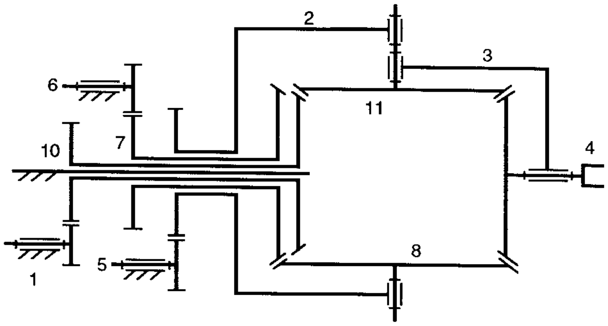

Figure 1. Functional representation of a wrist.

2. GEAR COUPLED MANIPULATORS

Figure 1 shows the functional representation of a GCM resembling a wrist mechanism. It has three DOF; links 5, 6, and 9 are the input links and link 4 is the output link, called the end-effector. Motors can be attached to the input links to drive the wrist. Figure 2 shows the associated canonical graph representation[1]. From the canonical graph representation, it can be seen that links 1, 2, 3, and 4 are the primary links, and links 5, 6, 7, 8, 9, 10, and 11 are the secondary links. Secondary links 5, 6, 7, 9, and 10 are carried by primary link 1, while secondary links 8 and 11 are carried by primary link 2. Figure 3 shows the EOLC of the wrist. A coordinate system is attached to each primary link of the EOLC in accordance with the Hartenberg and Denavit convention[5] where d_i is the translational distance along $\mathbf{Z}_i$-axis, and a_i and α_i are the offset distance and the twist angle between $\mathbf{Z}_i$ and $\mathbf{Z}_{i+1}$ axes, respectively. The angle $q_{i+1,i}$, measured from $\mathbf{X}_i$-axis to $\mathbf{X}_{i+1}$-axis about $\mathbf{Z}_i$-axis, is referred to as the joint angle, θ_i, for the EOLC. Note that for the wrist shown in Fig. 1, $\alpha_1 = \pi/2$, $\alpha_2 = -\pi/2$, $\alpha_3 = 0$, and $a_1 = a_2 = a_3 = 0$, respectively.

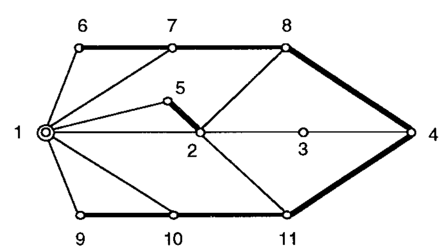

Figure 2. Canonical graph representation.

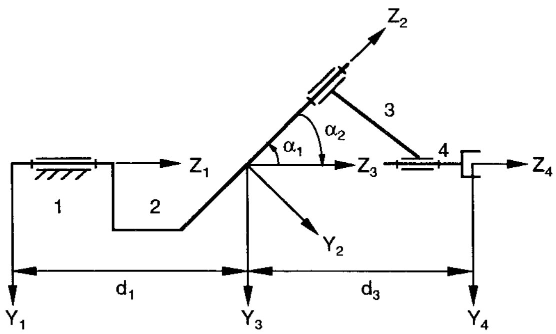

Figure 3. Typical EOLC of a three-DOF wrist.

Let Φ, Θ, and $\mathbf{X}$ be the displacement vectors associated with the actuators, joints, and the end-effector. Let $\dot{\Phi}$, $\dot{\Theta}$ and $\dot{\mathbf{X}}$ be the time derivatives of Φ, Θ, and $\mathbf{X}$. And let ξ, τ, and $\mathbf{F}$ be the generalized force vectors in the actuator-space, joint-space, and end-effector-space, respectively. The velocity vectors at the joint-space and end-effector-space are related by the Jacobian matrix, $\mathbf{J}$, as

$$\dot{\mathbf{X}} = \mathbf{J}\dot{\Theta} \tag{1}$$

and the torque vectors at the joint-space and end-effector-space are related by

$$\tau = \mathbf{J}^T\mathbf{F} \tag{2}$$

where $(\ )^T$ denotes the transpose of ( ).

The velocity vectors at actuator-space and joint-space can be related by using fundamental circuit theory and coaxial conditions[1,3] as

$$\dot{\Phi} = \mathbf{A}^T\dot{\Theta} \tag{3}$$

and the torque vectors at the joint-space and actuator-space are related by

$$\tau = \mathbf{A}\xi \tag{4}$$

where $\mathbf{A}$ is the structure matrix[3] and elements of $\mathbf{A}$ are functions of gear ratios.

For the wrist shown in Figure 1, the Jacobian matrix can be written as

$$\mathbf{J} = \begin{bmatrix} 0 & S_{\alpha_1}S_2 & S_{\alpha_1}C_{\alpha_2}S_2 + C_{\alpha_1}S_{\alpha_2}S_2C_3 + S_{\alpha_2}C_2S_3 \\ 0 & -S_{\alpha_1}C_2 & -S_{\alpha_1}C_{\alpha_2}C_2 - C_{\alpha_1}S_{\alpha_2}C_2C_3 + S_{\alpha_2}S_2S_3 \\ 1 & C_{\alpha_1} & C_{\alpha_1}C_{\alpha_2} - S_{\alpha_1}S_{\alpha_2}C_3 \end{bmatrix} \tag{5}$$

The torque vectors at the joint-space and actuator-space are related as

$$\begin{bmatrix} \tau_{21} \\ \tau_{32} \\ \tau_{43} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \xi_5 \\ \xi_6 \\ \xi_9 \end{bmatrix} \tag{6}$$

where τ_{21}, τ_{32}, and τ_{43} are the joint torques, ξ_5, ξ_6, and ξ_9 are the torques applied at input link 5, 6, and 9, respectively, and

$$\mathbf{A} = \begin{bmatrix} r_{2,5} & r_{7,6} & r_{10,9} \\ 0 & r_{7,6}r_{8,7} & r_{10,9}r_{11,10} \\ 0 & r_{7,6}r_{8,7}r_{4,8} & r_{10,9}r_{11,10}r_{4,11} \end{bmatrix} \tag{7}$$

where $r_{i,j} = N_i/N_j$ and N_i denotes the number of teeth on gear i, $r_{i,j} = +N_i/N_j$ if a positive rotation of gear i, with respect to the arm k of gear pair (i, j), provides a positive rotation of gear j. In addition, $r_{i,j} = -N_i/N_j$ if a positive rotation of gear i, with respect to the arm k of gear pair (i, j), provides a negative rotation of gear j, and where the sense of rotation is defined by applying the right-hand-screw rule to the $\mathbf{Z}_k$-axis.

From equation (6), it can be seen that the i-th row of the structure matrix $\mathbf{A}$ describes how the resultant torque at joint "i" is affected by the input actuators. On the other hand, the kth column of matrix $\mathbf{A}$ describes how the torque of an input actuator "k" is transmitted to various joints of a mechanism. Thus, the (i, k) element of structure matrix $\mathbf{A}$ represents how the torque of input actuator "k" is transmitted to joint "i" of the mechanism. Since the removal of all the gears from the mechanism results in an open-loop chain, torques can only be transmitted by gear trains except for the direct drive joint, and the joint torques affected by an actuator must be consecutive. The gear train that results in a series of nonzero elements in the kth column of structure matrix $\mathbf{A}$ is called the mechanical power transmission line for the input actuator k[3]. Note that the first nonzero element in the mechanical power transmission line for the input actuator k indicates the location of the input actuator. Let joint "i" be the first joint affected by input actuator k and $a_{i,k}$ be the first nonzero element of the mechanical power transmission line for the input actuator k. Let $(b_{j,i})_k$ be the ratio between the torque contributed by input actuator "k" at joint "j" and that a joint "i". The (j, k) element of structure matrix $\mathbf{A}$, $a_{i,k}$, can be written as

$$a_{j,k} = a_{i,k}(b_{j,i})_k \tag{8}$$

Note that $(b_{j,i})_k$ is equal zero if joint j is not affected by the kth input actuator.

For the wrist shown in Figure 1, there are three transmission lines: $5 \rightarrow 2$, $6 \rightarrow 7 \rightarrow 8 \rightarrow 4$, and $9 \rightarrow 10 \rightarrow 11 \rightarrow 4$. From equations (7) and (8), the structure matrix can be rewritten as

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{1,2}(b_{2,1})_2 & a_{1,3}(b_{2,1})_3 \\ 0 & a_{1,2}(b_{3,1})_2 & a_{1,3}(b_{3,1})_3 \end{bmatrix} \tag{9}$$

where

$$a_{1,1} = r_{2,5} \tag{10a}$$

$$a_{1,2} = r_{7,6} \tag{10b}$$

$$a_{1,3} = r_{10,9} \tag{10c}$$

$$(b_{2,1})_2 = r_{8,7} \tag{10d}$$

$$(b_{3,1})_2 = (b_{2,1})_2 r_{4,8} \tag{10e}$$

$$(b_{2,1})_3 = r_{11,10} \tag{10f}$$

$$(b_{3,1})_3 = (b_{2,1})_3 r_{4,11} \tag{10g}$$

3. KINEMATIC ISOTROPIC CONDITION

The generalized velocity ratio (K_v), which is defined as the ratio of the quadratic norms of the velocity vectors in end-effector space and actuator space, can be represented in the joint-space as[4]

$$K_v^2 = \frac{\dot{\Theta}^T \mathbf{J}^T \mathbf{J} \dot{\Theta}}{\dot{\Theta}^T \mathbf{A}\mathbf{A}^T \dot{\Theta}} \tag{11}$$

The value of K_v depends on the position as well as direction of motion of the end-effector. The extreme values of K_v are the square roots of the eigenvalues of the following generalized eigenvalue problem[6]

$$(\mathbf{J}^T \mathbf{J})\dot{\Theta} = \lambda(\mathbf{A}\mathbf{A}^T)\dot{\Theta} \tag{12}$$

Hence, the eigenvalues of equation (12) completely characterize the kinematic performance of a GCM at a given end-effector position. The condition number can be defined as the ratio of the maximal and minimal eigenvalues of equation (12). The transformation is said to be isotropic if the condition number of the overall transformation is equal to one. Under this condition the generalized ratio, $K_v = \sqrt{\lambda}$, is independent of the direction of motion. The manipulator with directional insensitive generalized velocity ratio is said to possess kinematic isotropic property at a given end-effector position. It has been shown that for λ being an n-fold root of the characteristic equation of equation (12), the following proportional condition must be satisfied[4]

$$(\mathbf{A}\mathbf{A}^T)_{i,j} = \frac{1}{k_v^2} (\mathbf{J}^T \mathbf{J})_{i,j} \tag{13}$$

where $(\ )_{i,j}$ denotes the (i, j) element of the matrix enclosed in the parenthesis.

4. SELECTION OF DRIVE TRAIN CONFIGURATION

Equation (13) can be used for the design of GCMs. Supposing the kinematic structure of the EOLC has been selected from the geometric consideration, the problem now is to select the proper drive train configuration such that the kinematic isotropic condition can be achieved at a prescribed posture, and with the selected drive train configuration, to define the gear ratios. For an n-DOF GCM, equation (13) yields $n(n + 1)/2$ nonlinear equations. However, the number of unknowns in equation (13) depends on the location of input actuators and the transmission lines arrangement, i.e., the number of nonzero elements in the structure matrix. It is essential that the number of unknowns in the structure matrix of the chosen drive train configuration is not less than the number of equations. If the number of unknowns is less than the number of equations, then the kinematic isotropic condition is not achievable at the design reference point with the chosen drive train configuration. Thus, special geometric arrangement of the EOLC is required and/or a new design reference point should be recommended to yield an isotropic condition. If the number of unknowns is greater than the number of equations, then there exist some free choices among the mechanical power transmission lines. This leads to the following axiom:

Axiom 1: For an n-DOF GCM to possess kinematic isotropic property, only those structure matrices with the following characteristics can be chosen as possible drive train configurations if none of the elements in the product of Jacobian matrix $\mathbf{J}^T\mathbf{J}$ is equal to zero at a given design reference point: (1) the num-

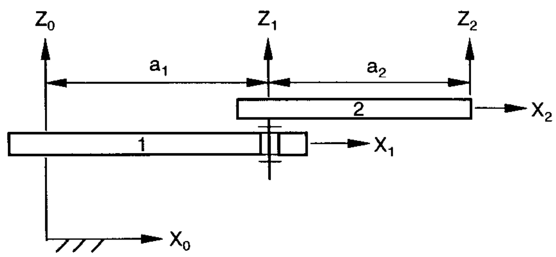

Figure 4. Typical EOLC of a two-DOF manipulator.

ber of nonzero elements in the structure matrix $\mathbf{A}$ is equal to or greater than $n(n+1)/2$; and (2) none of the elements in the product of structure matrix $\mathbf{AA}^T$ is zero.

In a case in which a certain element of the product of Jacobian matrix $\mathbf{J}^T\mathbf{J}$ is equal to zero at a given design reference point, the corresponding element in the product of structure matrix $\mathbf{AA}^T$ must vanish for equation (13) to be solvable. This leads to the following axiom:

Axiom 2: For an n-DOF GCM to possess kinematic isotropic property, only those structure matrices with the following characteristics can be chosen as possible drive train configurations if (i, j) element of the product of Jacobian matrix $\mathbf{J}^T\mathbf{J}$ is equal to zero at a given design reference point: (1) none of the mechanical power transmission lines in the structure matrix $\mathbf{A}$ has an effect on the (i, j) element of $\mathbf{AA}^T$; and (2) more than one mechanical power transmission line has an effects on the (i, j) element of the product of structure matrix $\mathbf{AA}^T$ but their overall effects are equal to zero. For the latter case, special gear ratio relations are required between the corresponding mechanical power transmission lines.

Figure 4 shows a typical EOLC of a two-DOF manipulator. The product of the Jacobian matrix can be written as

$$\mathbf{J}^T\mathbf{J} = \begin{bmatrix} a_1^2 + a_2^2 + 2a_1a_1\cos(\theta_2) & a_2[a_2 + a_1\cos(\theta_2)] \\ a_2[a_2 + a_1\cos(\theta_2)] & a_2^2 \end{bmatrix} \tag{14}$$

Figure 5 shows the four admissible structure matrices for two-DOF GCMs. It can be shown that if

$$\begin{bmatrix} \# & \# \\ \# & \# \end{bmatrix} \quad \begin{bmatrix} \# & \# \\ \# & 0 \end{bmatrix} \quad \begin{bmatrix} \# & 0 \\ \# & \# \end{bmatrix} \quad \begin{bmatrix} \# & 0 \\ 0 & \# \end{bmatrix}$$

g^2-1 $\quad$ g^2-2 $\quad$ gs-1 $\quad$ gs-2

Figure 5. Admissible structure matrices for two-DOF gear coupled manipulator.

none of the elements in the product of Jacobian matrix $\mathbf{J}^T\mathbf{J}$ is equal to zero at a given design reference point, only structure matrices g^2-1, g^2-2, and $gs-1$ can be chosen to achieve kinematic isotropic condition. On the other hand, if the design reference point is chosen at $\theta_2 = \cos^{-1}(-a_2/a_1)$, then the off-diagonal term in equation (14) is equal to zero and only structure matrices g^2-1 and $gs-2$ can be chosen to achieve kinematic isotropic condition.

For the three-DOF wrist shown in Figure 1, from equation (5), the product of Jacobian matrix can be written as

$$\mathbf{J}^T\mathbf{J} = \begin{bmatrix} 1 & C_{\alpha 1} & C_{\alpha 1}C_{\alpha 2} - C_2S_{\alpha 1}S_{\alpha 2} \\ C_{\alpha 1} & 1 & C_{\alpha 2} \\ C_{\alpha 1}C_{\alpha 2} - C_2S_{\alpha 1}S_{\alpha 2} & C_{\alpha 2} & 1 \end{bmatrix} \tag{15}$$

Figure 6 shows the thirty admissible structure matrices for three-DOF GCMs[3]. In Figures 5 and 6, matrices are arranged according to the location of the actuators[3]. The letters "g", "s", and "e" denote the location of the input actuators at the 1st, 2nd, and 3rd joint axes, respectively, which correspond to the ground, shoulder, and elbow joints of a robot arm. The power stands for the number of actuators installed on that joint axes. Figure 7 shows the fifteen structure matrices with which kinematic isotropic conditions can be obtained for three-DOF GCMs if none of the elements in the product of Jacobian matrix $\mathbf{J}^T\mathbf{J}$ is equal to zero at a given design reference point. Note that although the number of nonzero elements in structure matrices g^2s-6 and gs^2-3 are equal to six, they can not yield kinematic isotropic condition if the (1, 3) element in $\mathbf{J}^T\mathbf{J}$ is not equal to zero at a given design reference point, since the (1, 3) element in the $\mathbf{AA}^T$ is equal to zero. Table I shows the structure matrices with which kinematic isotropic condition can be obtained if certain elements in the product of Jacobian matrix $\mathbf{J}^T\mathbf{J}$ is equal to zero.

Hence, Axioms 1 and 2 provide a rational way to select drive train configurations for GCMs such that kinematic isotropic property can be made according to the predetermined geometric characteristics of the EOLC. With the form of available structure matrices, proper drive train configuration(s) can then be selected in the preference of actuator locations and mechanical simplicity of the mechanical power transmission lines.

$$\underset{g^3\text{-}1}{\begin{bmatrix} \# & \# & \# \\ \# & \# & \# \\ \# & \# & \# \end{bmatrix}} \quad \underset{g^3\text{-}2}{\begin{bmatrix} \# & \# & \# \\ \# & \# & \# \\ 0 & \# & \# \end{bmatrix}} \quad \underset{g^3\text{-}3}{\begin{bmatrix} \# & \# & \# \\ 0 & \# & \# \\ 0 & \# & \# \end{bmatrix}} \quad \underset{g^3\text{-}4}{\begin{bmatrix} \# & \# & \# \\ \# & \# & \# \\ 0 & 0 & \# \end{bmatrix}} \quad \underset{g^3\text{-}5}{\begin{bmatrix} \# & \# & \# \\ 0 & \# & \# \\ 0 & 0 & \# \end{bmatrix}}$$

$$\underset{g^2s\text{-}1}{\begin{bmatrix} \# & \# & 0 \\ \# & \# & \# \\ \# & \# & \# \end{bmatrix}} \quad \underset{g^2s\text{-}2}{\begin{bmatrix} \# & \# & 0 \\ \# & \# & \# \\ \# & \# & 0 \end{bmatrix}} \quad \underset{g^2s\text{-}3}{\begin{bmatrix} \# & \# & 0 \\ \# & \# & \# \\ \# & 0 & \# \end{bmatrix}} \quad \underset{g^2s\text{-}4}{\begin{bmatrix} \# & \# & 0 \\ \# & \# & \# \\ \# & 0 & 0 \end{bmatrix}} \quad \underset{g^2s\text{-}5}{\begin{bmatrix} \# & \# & 0 \\ \# & 0 & \# \\ \# & 0 & \# \end{bmatrix}} \quad \underset{g^2s\text{-}6}{\begin{bmatrix} \# & \# & 0 \\ \# & \# & \# \\ 0 & 0 & \# \end{bmatrix}}$$

$$\underset{g^2s\text{-}7}{\begin{bmatrix} \# & \# & 0 \\ \# & 0 & \# \\ \# & 0 & 0 \end{bmatrix}} \quad \underset{g^2s\text{-}8}{\begin{bmatrix} \# & \# & 0 \\ \# & 0 & \# \\ 0 & 0 & \# \end{bmatrix}}$$

$$\underset{g^2e\text{-}1}{\begin{bmatrix} \# & \# & 0 \\ \# & \# & 0 \\ \# & \# & \# \end{bmatrix}} \quad \underset{g^2e\text{-}2}{\begin{bmatrix} \# & \# & 0 \\ \# & \# & 0 \\ \# & 0 & \# \end{bmatrix}} \quad \underset{g^2e\text{-}3}{\begin{bmatrix} \# & \# & 0 \\ \# & 0 & 0 \\ \# & 0 & \# \end{bmatrix}} \quad \underset{g^2e\text{-}4}{\begin{bmatrix} \# & \# & 0 \\ \# & \# & 0 \\ 0 & 0 & \# \end{bmatrix}} \quad \underset{g^2e\text{-}5}{\begin{bmatrix} \# & \# & 0 \\ \# & 0 & 0 \\ 0 & 0 & \# \end{bmatrix}}$$

$$\underset{gs^2\text{-}1}{\begin{bmatrix} \# & 0 & 0 \\ \# & \# & \# \\ \# & \# & \# \end{bmatrix}} \quad \underset{gs^2\text{-}2}{\begin{bmatrix} \# & 0 & 0 \\ \# & \# & \# \\ \# & \# & 0 \end{bmatrix}} \quad \underset{gs^2\text{-}3}{\begin{bmatrix} \# & 0 & 0 \\ \# & \# & \# \\ 0 & \# & \# \end{bmatrix}} \quad \underset{gs^2\text{-}4}{\begin{bmatrix} \# & 0 & 0 \\ \# & \# & \# \\ 0 & 0 & \# \end{bmatrix}} \quad \underset{gs^2\text{-}5}{\begin{bmatrix} \# & 0 & 0 \\ 0 & \# & \# \\ 0 & \# & \# \end{bmatrix}} \quad \underset{gs^2\text{-}6}{\begin{bmatrix} \# & 0 & 0 \\ 0 & \# & \# \\ 0 & 0 & \# \end{bmatrix}}$$

$$\underset{gse\text{-}1}{\begin{bmatrix} \# & 0 & 0 \\ \# & \# & 0 \\ \# & \# & \# \end{bmatrix}} \quad \underset{gse\text{-}2}{\begin{bmatrix} \# & 0 & 0 \\ \# & \# & 0 \\ \# & 0 & \# \end{bmatrix}} \quad \underset{gse\text{-}3}{\begin{bmatrix} \# & 0 & 0 \\ \# & \# & 0 \\ 0 & \# & \# \end{bmatrix}} \quad \underset{gse\text{-}4}{\begin{bmatrix} \# & 0 & 0 \\ \# & \# & 0 \\ 0 & 0 & \# \end{bmatrix}} \quad \underset{gse\text{-}5}{\begin{bmatrix} \# & 0 & 0 \\ 0 & \# & 0 \\ 0 & \# & \# \end{bmatrix}} \quad \underset{gse\text{-}6}{\begin{bmatrix} \# & 0 & 0 \\ 0 & \# & 0 \\ 0 & 0 & \# \end{bmatrix}}$$

Figure 6. Admissible structure matrices for three-DOF gear coupled manipulator.

$$\underset{g^3\text{-}1}{\begin{bmatrix} \# & \# & \# \\ \# & \# & \# \\ \# & \# & \# \end{bmatrix}} \quad \underset{g^3\text{-}2}{\begin{bmatrix} \# & \# & \# \\ \# & \# & \# \\ 0 & \# & \# \end{bmatrix}} \quad \underset{g^3\text{-}3}{\begin{bmatrix} \# & \# & \# \\ 0 & \# & \# \\ 0 & \# & \# \end{bmatrix}} \quad \underset{g^3\text{-}4}{\begin{bmatrix} \# & \# & \# \\ \# & \# & \# \\ 0 & 0 & \# \end{bmatrix}} \quad \underset{g^3\text{-}5}{\begin{bmatrix} \# & \# & \# \\ 0 & \# & \# \\ 0 & 0 & \# \end{bmatrix}}$$

$$\underset{g^2s\text{-}1}{\begin{bmatrix} \# & \# & 0 \\ \# & \# & \# \\ \# & \# & \# \end{bmatrix}} \quad \underset{g^2s\text{-}2}{\begin{bmatrix} \# & \# & 0 \\ \# & \# & \# \\ \# & \# & 0 \end{bmatrix}} \quad \underset{g^2s\text{-}3}{\begin{bmatrix} \# & \# & 0 \\ \# & \# & \# \\ \# & 0 & \# \end{bmatrix}} \quad \underset{g^2s\text{-}4}{\begin{bmatrix} \# & \# & 0 \\ \# & \# & \# \\ \# & 0 & 0 \end{bmatrix}} \quad \underset{g^2s\text{-}5}{\begin{bmatrix} \# & \# & 0 \\ \# & 0 & \# \\ \# & 0 & \# \end{bmatrix}}$$

$$\underset{g^2e\text{-}1}{\begin{bmatrix} \# & \# & 0 \\ \# & \# & 0 \\ \# & \# & \# \end{bmatrix}} \quad \underset{g^2e\text{-}2}{\begin{bmatrix} \# & \# & 0 \\ \# & \# & 0 \\ \# & 0 & \# \end{bmatrix}} \quad \underset{gs^2\text{-}1}{\begin{bmatrix} \# & 0 & 0 \\ \# & \# & \# \\ \# & \# & \# \end{bmatrix}} \quad \underset{gs^2\text{-}2}{\begin{bmatrix} \# & 0 & 0 \\ \# & \# & \# \\ \# & \# & 0 \end{bmatrix}} \quad \underset{gse\text{-}1}{\begin{bmatrix} \# & 0 & 0 \\ \# & \# & 0 \\ \# & \# & \# \end{bmatrix}}$$

Figure 7. Structure matrices for three-DOF gear coupled manipulators with nonzero $(\mathbf{J}^T\mathbf{J})_{i,j}$.

Table I. Structure matrices for three-DOF gear coupled manipulators with certain zero $(\mathbf{J}^T\mathbf{J})_{i,j}$.

Cases	$()_{12}=0$	$()_{13}=0$	$()_{23}=0$	$()_{12}=0$ $()_{13}=0$	$()_{12}=0$ $()_{23}=0$	$()_{13}=0$ $()_{23}=0$	$()_{12}=0$ $()_{13}=0$ $()_{23}=0$
g^3	g^3-1 g^3-2 g^3-3 g^3-4 g^3-5	g^3-1 g^3-2 g^3-3	g^3-1 g^3-2 g^3-3	g^3-1 g^3-2 g^3-3	g^3-1 g^3-2 g^3-3	g^3-1 g^3-2 g^3-3	g^3-1 g^3-2 g^3-3
g^2s	g^2s-1 g^2s-2 g^2s-3 g^2s-4	g^2s-1 g^2s-2 g^2s-6 g^2s-8	g^2s-1 g^2s-2 g^2s-3 g^2s-5	g^2s-1 g^2s-2 g^2s-6	g^2s-1 g^2s-2 g^2s-3	g^2s-1 g^2s-2	g^2s-1 g^2s-2
g^2e	g^2e-1 g^2e-2	g^2e-1	g^2e-1	g^2e-1	g^2e-1	g^2e-1 g^2e-4 g^2e-5	g^2e-1 g^2e-4
gs^2		gs^2-3 gs^2-4	gs^2-1 gs^2-2	gs^2-5 gs^2-6			
gse		gse-3	gse-1	gse-5		gse-4	gse-6

$() = \mathbf{J}^T\mathbf{J}$

5. DRIVE TRAIN CHARACTERISTICS

From equation (13), structure matrix **A** determined from the kinematic isotropic condition for an n-DOF GCM can be written as

$$\mathbf{A} = K_v^{-1}\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \cdots & \varepsilon_{1n} \\ \varepsilon_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \varepsilon_{n1} & & \cdots & \varepsilon_{nn} \end{bmatrix} \tag{16}$$

where $\varepsilon_{i,j}$ is a function of the Jacobian matrix and can be determined from the kinematic isotropic condition. Hence, from equation (16), the train value in each mechanical power transmission line can be thought of as a product of two-stage gear reductions, i.e., K_v^{-1} and $\varepsilon_{i,j}$'s. This leads to the following axiom:

Axiom 3: For GCMs, kinematic isotropic condition can be made at a given design reference point through proper choice of drive train configuration. The train value for each mechanical power transmission line can be thought of as a product of two-stage gear reductions. The first-stage gear reduction, which is common to all mechanical power transmission lines, provides the overall reduction to the system. The second-stage gear reduction provides the necessary condition for a kinematic isotropic transformation.

From equation (8), the elements in the product of structure matrix $\mathbf{AA}^T$ contributed by input actuator "k" for an n-DOF GCM with n mechanical power transmission lines can be written as

$$(\mathbf{AA}^T)_{j,j} = \sum_{k=1}^{n} a_{i,k}^2 (b_{j,i}^2)_k \tag{17}$$

and

$$(\mathbf{AA}^T)_{j,m} = \sum_{k=1}^{n} a_{i,k}^2 (b_{j,i})_k (b_{m,i})_k \tag{18}$$

From equations (13), (17), and (18), it is clear that a sign change of the first nonzero element of the k-th mechanical power transmission line, $a_{i,k}$, does not change the kinematic isotropic condition. However, from equation (8), a sign change of the first nonzero element of the k-th mechanical power transmission line, $a_{i,k}$, results in sign changes of all nonzero elements in column k of structure matrix $\mathbf{A}$, accordingly. This leads to the following axiom:

Axiom 4: For n-DOF GCMs with n mechanical power transmission lines, kinematic isotropic condition can be made at a given design reference point through proper choice of gear ratios. The sign change of the first nonzero element of the k-th column of structure matrix $\mathbf{A}$ corresponds to a change of the direction of applied torque and will not change the kinematic isotropic condition. A sign change of the first nonzero element of the k-th column of structure matrix $\mathbf{A}$ results in sign change of the remaining nonzero elements in that column. Therefore, there are 2^n sets of solutions to equation (8) corresponding to the sign change of the first nonzero element of the n columns of a structure matrix $\mathbf{A}$.

6. OPTIMIZATION OF DYNAMIC PERFORMANCE

From equation (16), it can be shown that there are infinite sets of gear ratios that can be used in structure matrix $\mathbf{A}$ so that kinematic isotropic condition is satisfied if the generalized velocity ratio is treated as a scaling factor. This leaves additional room for the optimization of dynamic performance. Hence, this scaling factor can be used to optimize the acceleration capacity.

6.1. Acceleration Capacity

The equations of motion for an n-DOF GCM can be written in the joint-space as

$$\mathbf{M}\ddot{\Theta} + \dot{\Theta}^T\mathbf{C}\dot{\Theta} + \mathbf{G} = \mathbf{A}\xi \tag{19}$$

where $\mathbf{M}$ is an n by n inertia matrix, $\dot{\Theta}^T\mathbf{C}\dot{\Theta}$ is the generalized inertia force contributed by the Coriolis and centrifugal effects, and $\mathbf{G}$ is the generalized active force contributed by gravitational effect.

In formulating the generalized inertia forces of GCMs, Chen et al.[2] suggested the following approach. Firstly, all the secondary links are treated as being rigidly attached to their associated primary links and the generalized inertia forces due to the resultant equivalent open-loop chain are formulated. Secondly, the generalized inertia forces contributed by rotations of the secondary links with respect to their associated primary links are formulated. The inertia matrix of the second part of the generalized inertia forces can be considered as the sum of the inertia contribution due to input links and that due to all the other secondary links. However, the inertia contribution due to other secondary links can be neglected since they are usually one order of magnitude smaller than that due to the input links[4]. Thus, the inertia contribution due to input links dominates the inertia contribution of the second part of the generalized inertia forces. Let $\mathbf{M}_i$ be the inertia matrix of the second part of the generalized inertia forces due to input links. Chen and Tsai[7] showed that the inertia matrix $\mathbf{M}_i$ can be written as

$$\mathbf{M}_i = l_m\mathbf{A}\mathbf{U}\mathbf{A}^T \tag{20}$$

where

$$l_m = \left(\prod_{i=1}^{n} l_i\right)^{1/n} \tag{21}$$

and l_i is the inertia of the ith input link, $\mathbf{U}$ is a diagonal scaling matrix with its (i, i) element equal to l_i/l_m and its determinant equal to unity.

From equations (16) and (20), the inertia matrix $\mathbf{M}_i$ can be rewritten as

$$\mathbf{M}_i = K_v^{-2}\begin{bmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1n} \\ \delta_{12} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \delta_{1n} & & \cdots & \delta_{nn} \end{bmatrix} = K_v^{-2}\,\Delta \tag{22}$$

where $\delta_{i,j}$ is function of the inertia of input actuators and the second stage gear reduction $\varepsilon_{i,j}$'s. Taking determinant of both sides of equation (13), we have

$$\det(\mathbf{A}\mathbf{A}^T) = k_v^{-2n}\det(\mathbf{J}^T\mathbf{J}) \tag{23}$$

From Eqs. (20), (22), and (23), it can be shown that

$$\det(\Delta) = l_m^n\det(\mathbf{J}^T\mathbf{J}) \tag{24}$$

Let $\mathbf{M}_p$ be the inertia matrix of the first part of the generalized inertia forces, then the inertia matrix $\mathbf{M}$ can be written as

$$\mathbf{M} = \mathbf{M}_p + \mathbf{M}_i \tag{25}$$

For the cases in which the Coriolis, centrifugal and gravitational effects are insignificant, Chen and

Tsai[4] showed that acceleration capacity (A.C.) can be used as an index to indicate the ability of a manipulator to respond to a given set of input torques. The larger the acceleration capacity, the more responsive the system is. The acceleration capacity can be written as[4]

$$\text{A.C.} = \frac{[\det(\mathbf{J}^T\mathbf{J})\det(\mathbf{A}\mathbf{A}^T)]^{1/2}}{\det(\mathbf{M})} \tag{26}$$

Substituting equations (23) and (25) into equation (26) yields

$$\text{A.C.} = \det(\mathbf{J}^T\mathbf{J})Q^{-1} \tag{27}$$

where

$$Q = K_v^n \det(\mathbf{M}_p + \mathbf{M}_i) \tag{28}$$

Since the determinant of the product of Jacobian matrix, $\det(\mathbf{J}^T\mathbf{J})$, is a constant at a given end-effector position, acceleration capacity is a function of the generalized velocity ratio and the determinant of inertia matrix $\mathbf{M}$. Note that $\mathbf{M}_i$ is a function of gear ratios while $\mathbf{M}_p$ is a function of the joint angles and link mass/inertia properties. In what follows, we shall assume that adjusting gear ratios does not have a significant effect on the inertia matrix $\mathbf{M}_p$.

6.2. Two-DOF Manipulators

Let the inertia matrices $\mathbf{M}_p$ and $\mathbf{M}_i$ have the following form

$$\mathbf{M}_p = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{bmatrix} \tag{29}$$

$$\mathbf{M}_i = K_v^{-2}\begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{12} & \delta_{22} \end{bmatrix} = K_v^{-2}\,\Delta \tag{30}$$

Substituting equations (29) and (30) into equation (28) yields

$$Q = \det(\mathbf{M}_p)K_v^2 + \delta_{22}\rho_{11} - 2\delta_{12}\rho_{12} + \delta_{11}\rho_{22} + \det(\Delta)K_v^{-2} \tag{31}$$

Taking the derivative of equation (31) with respect to K_v and equating the resulting equation to zero, yields

$$K_v^4 = \frac{\det(\Delta)}{\det(\mathbf{M}_p)} \tag{32}$$

Substituting equation (24) in equation (32) we have

$$K_v^4 = \frac{l_m^2 \det(\mathbf{J}^T\mathbf{J})}{\det(\mathbf{M}_p)} \tag{33}$$

It can be shown that equation (33) provides the optimal condition for minimum Q since the sign of the second derivative of equation (31) with respect to K_v evaluated at the stationary point is always positive. Taking determinant of both sides of equation (22) and substituting the resulting equation into equation (32) yields

$$\det(\mathbf{M}_p) = \det(\mathbf{M}_i) \tag{34}$$

Hence, the optimum acceleration capacity can be reached when the determinant of inertia matrix contributed by input links is made equal to that contributed by primary links for two-DOF GCMs. From equation (33), it can be seen that the generalized velocity ratio is a function of inertia matrix $\mathbf{M}_p$ and the inertia of input liks at a given design reference point. Hence, the first stage gear reduction can be determined once the inertia of input links are selected with known $\mathbf{M}_p$ at a design reference point. On the other hand, the generalized velocity ratio can be adjusted to a desired value by choosing the proper inertia of input links.

Let $a_1 = 0.2286$ m, $a_2 = 0.1778$ m, and $\theta_2 = 112.885$ degrees; from equation (14), the product of the Jacobian matrix is

$$\mathbf{J}^T\mathbf{J} = \begin{bmatrix} 0.05226 & 0.01581 \\ 0.01581 & 0.03161 \end{bmatrix} \tag{35}$$

Let inertia of the input links be $8.79^* \, 10^{-6}$ kg − m^2 and $\mathbf{M}_p$ as

$$\mathbf{M}_p = \begin{bmatrix} 958 & 29.4 \\ 29.4 & 107 \end{bmatrix} (10^{-4}\,\text{kg} - \text{m}^2) \tag{36}$$

For the case structure matrix, $g^2 - 2$ is chosen as the drive train configuration, and it takes the following form:

$$\mathbf{A} = \frac{1}{k_v}\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{1,1}(b_{2,1})_1 & 0 \end{bmatrix} \tag{37}$$

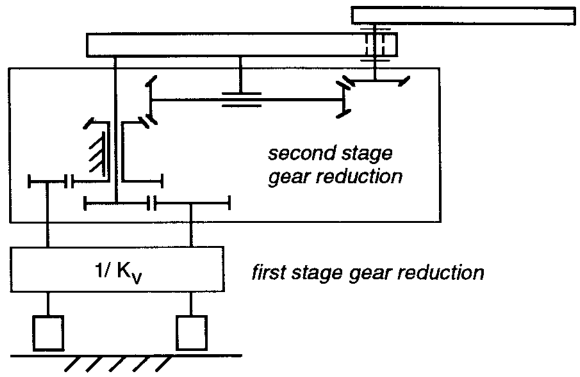

Figure 8. A two-DOF manipulator with optimum drive train configuration.

Then, from equations (33), (35), and (36), the generalized velocity ratio K_v can be obtained as 0.003128 (m) and $a_{1,1} = \pm 8.89$, $a_{1,2} = \pm 21.06$, and $(b_{2,1})_1 = 2$. Figure 8 shows one of the possible gearing configurations for the two-DOF manipulator.

6.3. Three-DOF Manipulators

Let the inertia matrices $\mathbf{M}_p$ and $\mathbf{M}_i$ have the following forms:

$$\mathbf{M}_p = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{12} & \rho_{22} & \rho_{23} \\ \rho_{13} & \rho_{23} & \rho_{33} \end{bmatrix} \tag{38}$$

$$\mathbf{M}_i = K_v^{-2} \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{12} & \delta_{22} & \delta_{23} \\ \delta_{13} & \delta_{23} & \delta_{33} \end{bmatrix} = K_v^{-2}\,\Delta \tag{39}$$

Substituting equations (38) and (39) into equation (25) and taking the determinant of the resulting equation yields

$$\det(\mathbf{M}) = \det(\mathbf{M}_p) + aK_v^{-2} + bK_v^{-4} + \det(\Delta)K_v^{-6} \tag{40}$$

where

$$\begin{aligned} a &= \rho_{11}\mathbf{M}_i(1,1) + \rho_{22}\mathbf{M}_i(2,2) + \rho_{33}\mathbf{M}_i(3,3) \\ &\quad + 2[\rho_{12}\mathbf{M}_i(1,2) + \rho_{13}\mathbf{M}_i(1,3) + \rho_{23}\mathbf{M}_i(2,3)] \end{aligned} \tag{41a}$$

$$\begin{aligned} b &= \delta_{11}\mathbf{M}_p(1,1) + \delta_{22}\mathbf{M}_p(2,2) + \delta_{33}\mathbf{M}_p(3,3) \\ &\quad + 2[\delta_{12}\mathbf{M}_p(1,2) + \delta_{13}\mathbf{M}_p(1,3) + \delta_{23}\mathbf{M}_p(2,3)] \end{aligned} \tag{41b}$$

and where $\mathbf{M}(i, j)$ denotes the cofactor of $\mathbf{M}$. Substituting equation (40) into equation (28) yields

$$Q = \det(\mathbf{M}_p)K_v^3 + aK_v + bK_v^{-1} + \det(\Delta)K_v^{-3} \tag{42}$$

Taking derivative of equation (42) with respect to k_v and equating the resulting equation to zero, we have

$$3\det(\mathbf{M}_p)K_v^2 + a - bK_v^{-2} - 3\det(\Delta)K_v^{-4} = 0 \tag{43}$$

Letting x be the square of K_v, equation (43) can be rewritten as

$$x^3 + \beta_1 x^2 + \beta_2 x + \beta_3 = 0 \tag{44}$$

where

$$\beta_1 = \frac{a}{3\det(\mathbf{M}_p)} \tag{45a}$$

$$\beta_2 = -\frac{b}{3\det(\mathbf{M}_p)} \tag{45b}$$

$$\beta_3 = -\frac{\det(\Delta)}{\det(\mathbf{M}_p)} \tag{45c}$$

Because matrices $\mathbf{M}_p$ and Δ are all positive definite, their determinants are all positive. Thus, from (45c), it can be shown that there exists a positive real root of equation (43). Taking the second derivative of equation (42) with respect to k_v, we have

$$\frac{\partial^2 Q}{\partial K_v^2} = 6\det(\mathbf{M}_p)K_v + 2bK_v^{-3} + 12\det(\Delta)K_v^{-5} \tag{46}$$

Subtracting K_v^{-2} times equation (43) from equation (40), we have

$$-2\det(\mathbf{M}_p) + 2bK_v^{-4} + 4\det(\Delta)K_v^{-6} \tag{47}$$

The sign of equation (47) is positive since the sign of equation (40) is always positive and the value of equation (43) is zero. Subtracting K_v times equation (47) from equation (46) yields

$$8\det(\mathbf{M}_p)K_v + 8\det(\Delta)K_v^{-5} > 0 \tag{48}$$

Thus, it is clear that the sign of equation (46), the second derivative of Q with respect to K_v, is always positive with a positive root of equation (43). Hence, an optimal generalized velocity ratio K_v can always be found for three-DOF GCMs such that the acceleration capacity is optimized.

Let the design reference point be chosen at $\alpha_1 = \pi/2$, $\alpha_2 = -\pi/2$ and $\theta_2 = 50$ degrees. From equation (15), the product of the Jacobian matrix is

$$\mathbf{J}^T\mathbf{J} = \begin{bmatrix} 1 & 0 & 0.6428 \\ 0 & 1 & 0 \\ 0.6428 & 0 & 1 \end{bmatrix} \tag{50}$$

Because the (1, 2) and (2, 3) elements of $\mathbf{J}^T\mathbf{J}$ are equal to zero, from Table I, there are seven structure matrices that can be selected as proper gear train configurations to achieve kinematic isotropic property. For the wrist mechanism, the g^3 series are preferred for their better dynamic performance from the view of actuators' locations. Assuming structure matrix g^3-3 is chosen, let it have the form of equation (9). Because the number of unknowns in structure matrix g^3-3 is equal to seven, one variable can be set as a free choice. Let $g = a_{1,3}/a_{1,2}$ as the free choice, $k_{13} = (\mathbf{J}^T\mathbf{J})_{1,3}$, the nonzero elements in the structure matrix $\mathbf{A}$ can be solved from isotropic condition and written as $a_{1,1} = \pm\sqrt{1-k_{13}^2}$, $a_{1,2} = \pm k_{13}/\sqrt{(1+g^2)}$, $a_{1,3} = \pm g k_{13}/\sqrt{(1+g^2)}$, $(b_{2,1})_2 = -g/k_{13}$, $(b_{2,1})_3 = 1/(gk_{13})$, $(b_{3,1})_2 = 1/k_{13}$, and $(b_{3,1})_3 = 1/k_{13}$. Note that the free choice g defines the special relation between the second and third mechanical power transmission lines.

Let inertia of the input links be $8.79*10^{-6}$ kg $-$ m^2 and $g = 1$, from equations (20) and (50), then the inertia matrix $\mathbf{M}_i$ can be written as

$$\mathbf{M}_i = \frac{1}{K_v^2}\begin{bmatrix} 0.0879 & 0 & 0.0565 \\ 0 & 0.0879 & 0 \\ 0.0565 & 0 & 0.0879 \end{bmatrix}(10^{-4}\,\mathrm{kg}-\mathrm{m}^2) \tag{51}$$

Assuming inertia matrix $\mathbf{M}_p$ as

$$\mathbf{M}_p = \begin{bmatrix} 42534.31 & 0.0 & 1513279 \\ 0.0 & 74922.9 & 0.0 \\ 1513279 & 0.0 & 15666.62 \end{bmatrix}(10^{-7}\,\mathrm{kg}-\mathrm{m}^2) \tag{52}$$

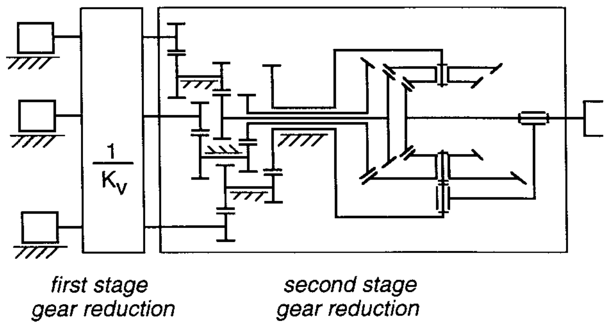

Figure 9. A wrist with optimum drive train configuration.

Then, the optimal generalized velocity ratio k_v can be obtained as 0.047566 (m). Figure 9 shows one of the possible drive train arrangements with the structure matrix written as

$$\mathbf{A} = 21.023\begin{bmatrix} 0.766 & 0.4545 & 0.4545 \\ 0 & -0.7071 & 0.7071 \\ 0 & 0.7071 & 0.7071 \end{bmatrix} \tag{53}$$

Note that for the case structure matrix, g^3-5 is chosen as the desired drive train configuration, which is commonly used in wrist mechanisms; kinematic isotropic property can not be achieved if the EOLC has the following geometric characteristics: $\alpha_1 = \pi/2$ and $\alpha_2 = -\pi/2$.

As the torque contribution ratio between joint i and j due to input actuator k is determined, the actual drive train arrangement of each mechanical power transmission line can be constructed accordingly. The arrangement of gears can be laid out according to the sign of nonzero elements of A. Note that for the cases where using several gear pairs is more desirable than using a single pair, and/or adjusting the center distance between rotational axes, idle gear(s) can be added. Several methods[8,9,10] have been developed to find the number of teeth for each gear train required to provide a specific ratio.

7. CONCLUSIONS

A methodology for the drive train configurations determination of GCMs is developed. It is shown that once the geometric characteristics of EOLC are specified, the manipulator can be designed to possess kinematic isotropic property with optimum dynamic performance through proper choice of the drive train configuration and gear ratios. This approach provides a rational procedure for the design of GCMs with desired kinematic and dynamic characteristics.

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