

## KINEMATIC ANALYSIS OF GEARED ROBOT MANIPULATORS BY THE CONCEPT OF STRUCTURAL DECOMPOSITION

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**Abstract**—A new and systematic methodology for the kinematic analysis of geared robot manipulators is presented. It is shown that, by decomposing the mechanical transmission lines as an input unit and several transmission units connected in series, the kinematic relation between the local input and output links of each unit can be systematically formulated. Besides, angular displacements of the connecting links, with respect to its associated primary link, can be symbolically derived in terms of joint angles by a unit-by-unit evaluation procedure. This unit-by-unit evaluation procedure provides a better insight of the effects of the input and transmission units on the torque and leads to an automated derivation of the kinematic relations without the need to solve a set of linear equations simultaneously. © 1998 Elsevier Science Ltd. All rights reserved

### INTRODUCTION

The kinematic structure of a robot manipulator often takes the form of an open-loop configuration. An open-loop robot manipulator is mechanically simple and easy to construct. However, it does require the actuators to be located along the joint axes which, in turn, degrades the dynamic performance of the system. For this reason, many robotic mechanisms are constructed in a partially closed-loop configuration to ease the actuator design and to reduce the inertia loads on the actuators. For the case of geared robot manipulators, gear trains are used to permit the actuators to be located as close to the base link as possible.

Various methods for the kinematic analysis of geared robot manipulators can be found in the literature [1–5]. Among them, the most promising approach is perhaps the systematic method introduced by Tsai [5]. The method utilizes the concept of fundamental circuit of geared mechanisms introduced by Freudenstein [6] and coaxial conditions between coaxial links to form the displacement equations. Then the relative rotation between every two adjacent links in the equivalent open-loop chain is derived by solving the set of displacement equations simultaneously. The concepts of fundamental circuit and coaxiality condition are powerful tools for the kinematic analysis of geared robot manipulators. However, the analysis involves the solution of a set of linear equations for all the kinematic variables. It does not provide much insight into the kinematic structure of a geared robot manipulator.

Chang and Tsai [7] showed that the kinematic structure of a geared robot manipulator can be described by an equivalent open-loop chain and mechanical transmission lines. Chen and Shieu [8] showed that a mechanical transmission line can be decomposed as an input unit and several transmission units connected in series. In this paper, the concept of input and transmission units will be applied for the kinematic analysis of geared robot manipulators. It will be shown that, by decomposing a geared robot manipulator into the equivalent open-loop chain and mechanical transmission lines and each mechanical transmission line into input and transmission units, the input displacements can be symbolically expressed in terms of the joint angles by a unit-by-unit backward evaluation. The main advantage of this method is that algebraic expression for the kinematic relation between the actuator-space and joint-space can be easily obtained without mathematically manipulating the set of linear equations. In addition, this unit-by-unit evaluation procedure provides a better insight of the effects of the input and transmission units on the torque transmission of each mechanical transmission line. This can be very

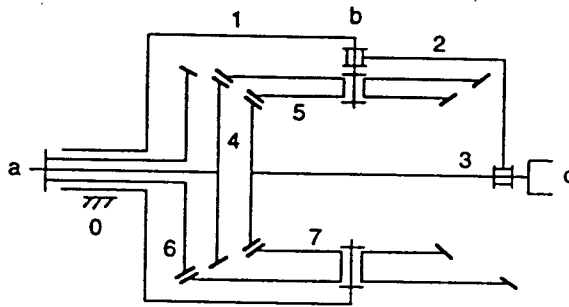


Fig. 1. A Three-DOF wrist mechanism.

helpful in the identification of kinematic behavior during the design phase of a geared robot manipulator.

**GEARED ROBOT MANIPULATORS**

The kinematic structure of a three degree-of-freedom (dof) geared robotic wrist is shown in Fig. 1, in which a number by itself represents the number of a link. In the mechanism, links 1, 4 and 6 are the input links and link 3 is the output link, called the end-effector. Actuators can be attached to links 1, 4 and 6 to drive the mechanism. Through spur or bevel gear trains, power is transmitted to end-effector.

In the graph representation, links are denoted by vertices, turning pairs by thin edges, gear pairs by heavy edges, and thin edges are labeled according to their axis locations in space. Figure 2 shows the graph representation of the wrist. Note that darkened vertices in Fig. 2 represent the input links. For geared mechanisms, it is well-known that the removal of all heavy edges from its graph representation results in a spanning tree. By re-arranging the coaxial thin edges in a spanning tree, a canonical spanning tree can be obtained so that all thin-edge paths originating from the base link have distinct edge labels. Figure 3 shows the canonical spanning tree of the wrist. In the canonical spanning tree of a geared robot manipulator, linkage made up of links and joints on a thin-edge path originating from the base link and ending at a leaf [9] represents an open-loop chain. Among these open-loop chains, the one that starts from the base link and ends at the output link is defined as the equivalent open-loop chain [5]. Each link in the equivalent open-loop chain is referred to as a primary link, and all other links are called the secondary links [10]. Figure 4 shows the equivalent open-loop chain of the wrist.

Beginning from the base link, primary links are numbered sequentially from 0 to  $n$  for an  $n$ -dof geared robot manipulator. To facilitate the analysis, a coordinate system can be attached to each primary link of the equivalent open-loop chain in accordance with the Denavit-Hartenberg [11] where  $d_{i,i-1}$  is the translational distance along  $Z_{i-1}$ -axis,  $a_{i,i-1}$  and  $\alpha_{i,i-1}$  are the offset distance and the twist angle between  $Z_{i-1}$  and  $Z_i$  axes, respectively. The angle  $q_{i,i-1}$  measured from  $X_{i-1}$ -axis to  $X_i$ -axis about  $Z_{i-1}$ -axis is referred to as the joint angle,  $\theta_i$ , for the equivalent open-loop chain.

The distance between two vertices is defined as the length of the thin-edge path connecting them in the canonical spanning tree. For the case that the distance between a secondary link  $j$

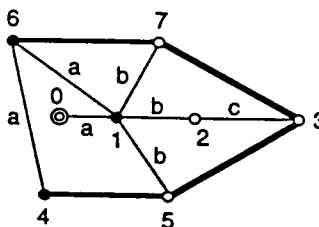


Fig. 2. Graph representation.

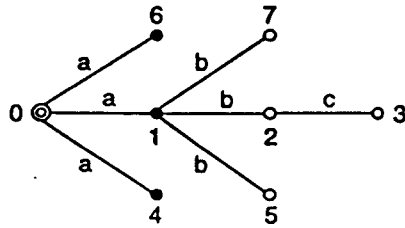


Fig. 3. Canonical spanning tree.

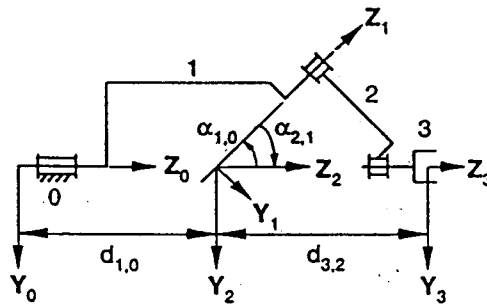


Fig. 4. The equivalent open-loop chain.

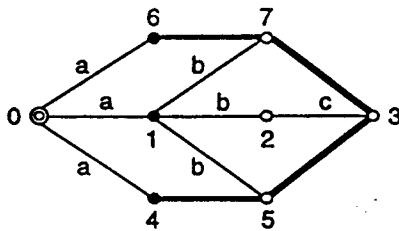


Fig. 5. Canonical graph representation.

and a primary link  $i$  is equal to one, it is said that primary link  $i$  carries secondary link  $j$  while secondary link  $j$  is associated with primary link  $i$ . From Fig. 3, it can be seen that links 0, 1, 2 and 3 are the primary links, links 4, 5, 6, and 7 are the secondary links. Primary link 0 carries secondary links 4, and 6, while primary link 1 carries secondary links 5 and 7. Also, secondary links 4 and 6 are associated with primary link 0, while secondary links 5 and 7 are associated with primary link 1. Note that a graph representation of a geared robot manipulator is called its canonical graph representation [5] if the canonical spanning tree can be obtained by deleting its heavy edges. Figure 5 shows the canonical graph representation of the wrist mechanism.

### MECHANICAL TRANSMISSION LINES

The arrangement of the secondary links in a geared robot manipulator, which describes where the input actuators are located and how the input torques are transmitted to various joints of the mechanism, forms the mechanical transmission line [7, 12]. Through mechanical transmission lines, which consist of spur or bevel gear trains, torques are transmitted to the end-effector on which payload can be attached. By collecting the vertices and edges associated with a mechanical transmission line, a graph which represents the mechanical transmission line can be identified. Note that the graph representation of a mechanical transmission line is a subgraph of the graph of its associated geared robot manipulator. The length of a mechanical transmission line is defined as the number of joints in its graph representation. A mechanical transmission line is

called a direct drive mechanical transmission line if the input link is one of the primary links. Hence, a direct drive mechanical transmission line has length equal to one. Two mechanical transmission lines are said to be decoupled if they do not have any secondary link in common in their graph representation. In this paper, we will concentrate on the geared robot manipulators of independent joint motion and decoupled mechanical transmission lines with the following characteristics:

- (1) The number of mechanical transmission lines in a geared robot manipulator is equal to the number of dof of the mechanism;
- (2) Each mechanical transmission line contains an input link.

Fig. 6 shows the graph representation of the mechanical transmission lines of the example wrist mechanism. It can be seen that the mechanical transmission line in Fig. 6(a) is a direct drive mechanical transmission line which torque is exerted on primary link 1 directly. Both mechanical transmission lines shown in Fig. 6 (b) and (c) have length equal to three. For the mechanical transmission line corresponding to Fig. 6(b), torque exerted by secondary link 4 is transmitted to primary link 3 by gear pairs mounted on links 4 and 5, and links 5 and 3. For the mechanical transmission line corresponding to Fig. 6(c), torque exerted on secondary link 6 is transmitted to primary link 3 by gear pairs mounted on links 6 and 7, and links 7 and 3. From Fig. 6, it can be seen that decoupled mechanical transmission lines can have the equivalent open-loop chain or part of the equivalent open-loop chain in common.

Chen and Shieu [8] showed that, by re-arranging the coaxial links, a pseudo-isomorphic graph representation [13] of a mechanical transmission line can be obtained. Figure 7 shows the pseudo-isomorphic graphs of the non-direct drive mechanical transmission lines of Fig. 6. From the pseudo-isomorphic graph representation, they showed that a mechanical transmission line can be viewed as a collection of non-fractionated kinematic chains connected in series with the following characteristics:

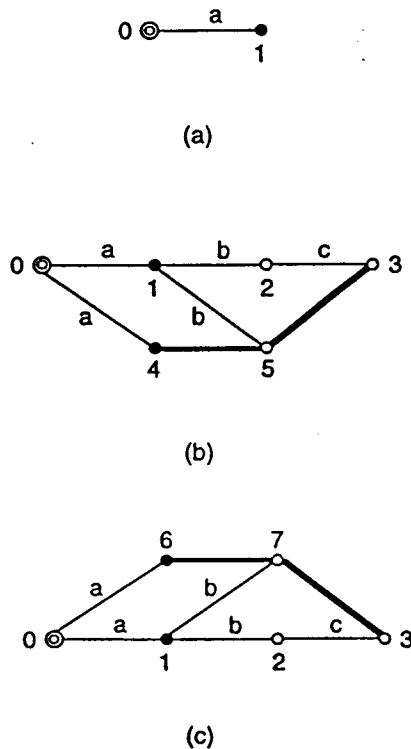
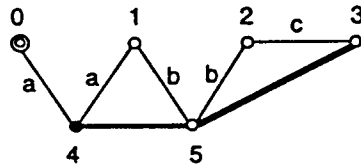
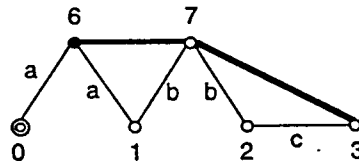


Fig. 6. Mechanical transmission lines of the wrist.



(a)



(b)

Fig. 7. Pseudo-isomorphic graphs.

(3) The first unit of a mechanical transmission line, which contains the input link, is called the input unit. The remainings are a series of geared kinematic chains (GKCs) called transmission units connected together.

(4) The number of input and transmission units of a mechanical transmission line is equal to the length of the mechanical transmission line.

(5) Each unit is connected to its adjacent unit by sharing a common secondary link called the connecting link. A preceding unit uses its post-connecting link to connect the pre-connecting link of its succeeding unit. The pre-connecting link of a unit is considered the local input to the unit while the post-connecting link is considered the local output to the unit.

(6) For the two-link chain, there is only one connecting link and the connecting link is the input link.

(7) The revolute joint between the connecting link and the primary link of a unit is coaxial with that of its associated adjacent unit. Hence, by re-arranging the coaxial revolute joints at the connecting links and primary links of each set of adjacent units, an articulated joint of the equivalent open-loop chain can be formed.

(8) By assigning the post-connecting link of a transmission unit as a primary link, the transmission unit can be used as the last unit of a mechanical transmission line. The joint between the primary links forms an articulated joint, which is a part of the equivalent open-loop chain. The articulated joint is the last joint influenced by the associated mechanical transmission line.

It can be seen that, for the mechanical transmission line corresponding to Fig. 7(a), the two-link chain (0, 4) acts as the input unit and the three-link GKCs (1, 4, 5) and (2, 5, 3) act as the transmission units. Secondary link 4 acts as the connecting link between input unit (0, 4) and transmission unit (1, 4, 5) while secondary link 5 acts as the connecting link between transmission units (1, 4, 5) and (2, 5, 3). By re-arranging the coaxial links at axis-*a*, an articulated joint can be formed between primary links 0 and 1. By re-arranging the coaxial links at axis-*b*, an articulated joint can be formed between primary links 1 and 2. Also, for the mechanical transmission line corresponding to Fig. 7(b), secondary link 6, as a connecting link, connects the two-link input unit (0, 6) and three-link GKC transmission unit (1, 6, 7), while secondary link 7, as a connecting link, connects the three-link GKC transmission units (1, 6, 7) and (2, 7, 3). By re-arranging the coaxial links at axis-*a* and axis-*b* of the mechanical transmission line, articulated joints can be formed respectively between primary links 0 and 1 at axis-*a* and between primary links 1 and 2 at axis-*b*.

Chen and Shieu [8] established the following characteristics for the identification of input and transmission units of a mechanical transmission line:

(9) In each unit, one and only one link can be assigned as the primary link. The other links are classified as secondary links.

(10) In each unit, a secondary link which is incident with the primary link can be assigned as a connecting link. There are two connecting links in each unit except for the two-link chain case.

(11) In an input unit, connecting links are coaxial with the primary link.

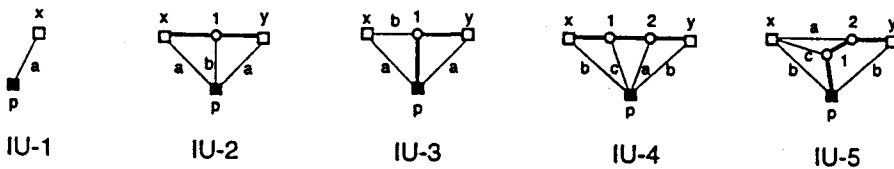
(12) In an input unit, one connecting link can be assigned as the input link.

(13) In a transmission unit, the connecting links and the primary link form a thin-edge path with two distinct edge labels.

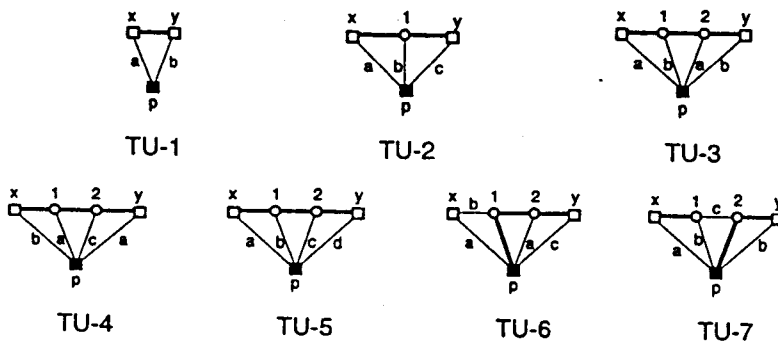
(14) In each unit, if a connecting link is assigned as a pre-connecting link, then the other connecting link will be the post-connecting link.

(15) There is no redundant link in each unit. A link is considered a redundant link if the removal of it and its associated joint does not change the torque transmission relation between the two connecting links.

Chen and Shieu[8] showed that, with the characteristics of input and transmission units, admissible input and transmission units can be systematically identified from the atlas of non-isomorphic GKC's enumerated by [6, 13-17]. Figure 8 shows the admissible input and transmission units up to five links [5]. In Fig. 8, IU stands for the input unit and TU stands for the transmission unit. Also, in Fig. 8, a secondary link is represented by a circle node, a primary link  $p$  is represented by a solid rectangular node, and a connecting link  $x$  or  $y$  is represented by a rectangular node, respectively. Note that the two-link input unit (IU-1) and three-link GKC transmission unit (TU-1) shown in Fig. 8 are the simplest forms of the admissible units. Since admissible input and transmission units, except IU-1, are identified from the GKC's, they are called geared units.



(a). Input Units



(b). Transmission Units

■ : primary link,    □ : connecting link,    ○ : secondary link

Fig. 8. Admissible input and transmission units

## SPEED RATIO OF UNIT

Let  $i, j$  and  $k$  be three coaxial links, then the relative angular displacements among these three links can be related as:

$$q_{i,k} = q_{i,j} + q_{j,k} \quad (1)$$

where  $q_{i,j}$  denotes the relative angular displacements of link  $i$  with respect to link  $j$ .

The kinematics of an epicyclic gear train can be analyzed by any of the well-known methods, such as the relative velocity method [18, 19], the energy method [20], the bond graph method [21], the vector-loop method [22–24], the signal graph method [4] and the method of fundamental circuit equation [6]. In the following, the method of fundamental circuit equation is employed. Let links  $j$  and  $k$  be a gear pair and  $i$  be the carrier. Then, links  $i, j$  and  $k$  form a fundamental circuit and a fundamental circuit equation can be written as:

$$q_{j,i} = e_{kj}q_{k,i} \quad (2)$$

where  $e_{kj} = \pm N_k/N_j$ , denotes the gear ratio for the gears mounted on links  $k$  and  $j$ , the positive or negative value varies according as a positive rotation of gear  $k$  results in a positive or negative rotation of gear  $j$  about their pre-defined axes of rotation, and  $N_k$  is the teeth number of gear  $k$ .

For units having more than one gear pair, fundamental circuit equation, such as Equation (2), can be set up according to the number of gear pairs in the unit. For example, the fundamental circuit equations of TU-7 in Fig. 8 can be written as

$$q_{x,p} = e_{1x}q_{1,p} \quad (3)$$

$$q_{1,y} = -q_{y,1} = -e_{2y}q_{2,1} \quad (4)$$

$$q_{2,1} = e_{p2}q_{p,1} = -e_{p2}q_{1,p} \quad (5)$$

Since links  $y, p$  and  $1$  are coaxial, from Equation (1), we have

$$q_{1,p} = q_{1,y} + q_{y,p} \quad (6)$$

Substituting Equation (5) in Equation (4) yields

$$q_{1,y} = e_{p2}e_{2y}q_{1,p} \quad (7)$$

Substituting Equation (7) in Equation (6), we have

$$q_{1,p} = \frac{1}{(1 - e_{p2}e_{2y})} q_{y,p} \quad (8)$$

Let link  $x$  be the pre-connecting link and link  $y$  be the post-connecting link. By substituting Equation (8) in Equation (3), the angular displacement between the local input  $x$  and primary link  $p$ ,  $q_{x,p}$ , can be represented as a function of the angular displacement between local output  $y$  and primary link  $p$ ,  $q_{y,p}$ , as

$$q_{x,p} = \frac{e_{1x}}{(1 - e_{p2}e_{2y})} q_{y,p} = G_f q_{y,p} \quad (9)$$

where  $G_f$  is called the forward gain associated with the TU-7 unit.


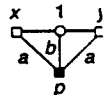
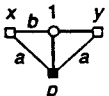
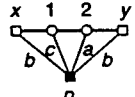
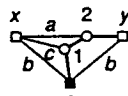
For the case that link  $y$  is chosen as the local input and link  $x$  as local output, from Equation (9), angular displacement between  $q_{y,p}$  and  $q_{x,p}$  can be represented as

$$q_{y,p} = \frac{(1 - e_{p2}e_{2y})}{e_{1x}} q_{x,p} = G_b q_{x,p} \quad (10)$$

where  $G_b$  is called the backward gain associated with the TU-7 unit.

Note that the number of fundamental circuit equations and coaxial equation(s) associated with the admissible geared units shown in Fig. 8 is fairly limited. Hence, the angular displace-

Table 1. Forward and backward gains of input units.

	$G_f$	$G_b$
 <p>IU-1</p>	1	1
 <p>IU-2</p>	$e_{y1}e_{1,x}$	$e_{x1}e_{1,y}$
 <p>IU-3</p>	$\frac{1}{1 - e_{p1}e_{1,y}}$	$1 - e_{p1}e_{1,y}$
 <p>IU-4</p>	$e_{y2}e_{21}e_{1,x}$	$e_{x1}e_{12}e_{2,y}$
 <p>IU-5</p>	$\frac{1}{1 - e_{p1}e_{12}e_{2,y}}$	$1 - e_{p1}e_{12}e_{2,y}$

ment relation between the connecting links and the primary link of the admissible geared units can be easily derived. Tables 1 and 2, respectively, show the forward gain and backward gain associated with the admissible input and transmission units. Also note that for the two-link input unit, the forward and backward gains are equal to one.

PROPAGATION BETWEEN UNITS

Figure 9 shows a typical mechanical transmission line with length equal to  $k$ . In Fig. 9, an input unit is connected with  $k - 1$  transmission units. The connecting links are numbered as  $j$  and  $j + 1, \dots, j + k$ , and the primary links are numbered as  $i$  and  $i + 1, \dots, i + k - 1$  for the input unit and transmission units, respectively. The rotation of connecting link  $j$  with respect to primary link  $i$ ,  $q_{j,i}$ , is considered the input angular displacement. For the first unit, the angular displacement relation between  $q_{j,i}$  and  $q_{j+1,i}$ , from Equation (9) and Table 1, can be written as

$$q_{j,i} = G_{j,1}q_{j+1,i} \tag{11}$$

where  $G_{j,i}$  denotes the forward or backward gain of the  $i$ -th unit of mechanical transmission line with connecting link  $j$  as input link.

For the second unit of the mechanical transmission line (i.e. the first transmission unit), we have

$$q_{j+1,i+1} = G_{j,2}q_{j+2,i+1} \tag{12}$$

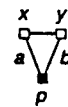
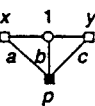
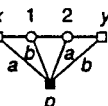
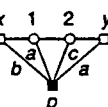
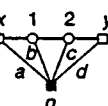
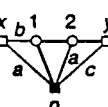
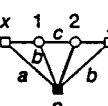
The coaxial condition between the first two units, from Equation (1), can be written as

$$q_{j+1,i} = q_{j+1,i+1} + q_{i+1,i} = q_{j+1,i+1} + \theta_{i+1} \tag{13}$$

By substituting Equation (12) into Equation (13), the angular displacement of the post-connecting link of the first unit, link  $j + 1$ , with respect to its associated primary link, link  $i$ , can be written as



Table 2. Forward and backward gains of transmission units.

	$G_f$	$G_b$
 <p>TU-1</p>	$e_{y,x}$	$e_{x,y}$
 <p>TU-2</p>	$e_{y1}e_{1,x}$	$e_{x1}e_{1,y}$
 <p>TU-3</p>	$e_{y2}e_{21}e_{1,x}$	$e_{x1}e_{12}e_{2,y}$
 <p>TU-4</p>	$e_{y2}e_{21}e_{1,x}$	$e_{x1}e_{12}e_{2,y}$
 <p>TU-5</p>	$e_{y2}e_{21}e_{1,x}$	$e_{x1}e_{12}e_{2,y}$
 <p>TU-6</p>	$\frac{e_{y2}}{1 - e_{p1}e_{12}}$	$(1 - e_{p1}e_{12})e_{2,y}$
 <p>TU-7</p>	$\frac{e_{1,x}}{1 - e_{p2}e_{2,y}}$	$e_{x1}(1 - e_{p2}e_{2,y})$

$$q_{j+1,i} = \theta_{i+1} + G_{j,2}q_{j+2,i+1} \tag{14}$$

Similarly, the angular displacement of the post-connecting link of the  $(k - 2)$ -th unit, link  $j + k - 2$ , with respect to its associated primary link, link  $i + k - 3$ , can be written as

$$q_{j+k-2,i+k-3} = \theta_{j+k-2} + G_{j,k-1}q_{j+k-1,i+k-2} \tag{15}$$

And the angular displacement of the post-connecting link of the  $(k - 1)$ -th unit, link  $j + k - 1$ , with respect to its associated primary link, link  $i + k - 2$ , can be written as

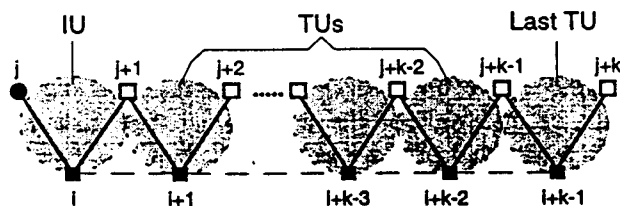
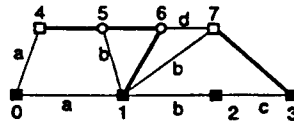
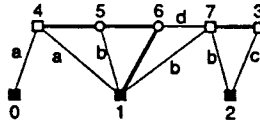


Fig. 9. A typical mechanical transmission line.



(a)



(b)

Fig. 10. A mechanical transmission line.

$$q_{j+k-1,i+k-2} = \theta_{i+k-1} + G_{j,k}q_{j+k,i+k-1} \tag{16}$$

From point (8) in the list, note that the post-connecting link  $j + k$  of the  $k$ -th transmission unit will be assigned as primary link  $i + k$  to form the last joint influenced by the mechanical transmission line. Hence,

$$q_{j+k,i+k-1} = q_{i+k,i+k-1} = \theta_{i+k} \tag{17}$$

Substituting Equation (17) into Equation (16) yields

$$q_{j+k-1,i+k-2} = \theta_{i+k-1} + G_{j,k}\theta_{i+k} \tag{18}$$

By substituting Equation (18) into Equation (15), we have

$$\begin{aligned} q_{j+k-2,i+k-3} &= \theta_{i+k-2} + G_{j,k-1}[\theta_{i+k-1} + G_{j,k}\theta_{i+k}] \\ &= \theta_{i+k-2} + \sum_{m=i+k-1}^{i+k} \left[ \left( \prod_{n=k-1}^{m-1} G_{j,n} \right) \theta_m \right] \end{aligned} \tag{19}$$

From Equation (19), it can be shown that angular displacement of the post connecting link of the  $g$ -th unit, link  $z$ , with respect to its primary link, link  $p$ , of a mechanical transmission line with length equal to  $k$ , can be written as

$$q_{z,p} = \theta_{p+1} + \sum_{m=p+2}^{p+(k-g)+1} \left[ \left( \prod_{n=g+1}^{m-(p-g+1)} G_{j,n} \right) \theta_m \right] \tag{20}$$

From Equation (20), the angular displacement of the post-connecting link of the second unit of the mechanical transmission line corresponding to Fig. 7(a) with respect to its associated primary link 1,  $q_{5,1}$ , can be written as

$$q_{5,1} = \theta_2 + G_{4,3}\theta_3 = \theta_2 + e_{35}\theta_3 \tag{21}$$

Similarly, from Tables 1 and 2, we have  $G_{6,3} = G_{(T)TU-1} = e_{37}$ , the angular displacement of the post-connecting link of the second unit of the mechanical transmission line corresponding to Fig. 7(b) with respect to its associated primary link 1,  $q_{7,1}$ , can be written as

$$q_{7,1} = \theta_2 + G_{6,3}\theta_3 = \theta_2 + e_{37}\theta_3 \tag{22}$$

From Equation (20), Equation (14) can be rewritten as

$$q_{j+1,i} = \theta_{i+1} + \sum_{m=i+2}^{j+k} \left[ \left( \prod_{n=2}^{m-1} G_{j,n} \right) \theta_m \right] \quad (23)$$

By substituting Equation (23) into Equation (11), the input angular displacement,  $q_{j,i}$ , of a  $k$ -length mechanical transmission line can be written as function of joint angles:

$$q_{j,i} = \sum_{m=i+1}^{i+k} \left[ \left( \prod_{n=1}^{m-1} G_{j,n} \right) \theta_m \right] \quad (24)$$

Fig. 10(a) shows a mechanical transmission line with IU-1 as input unit and TU-6, TU-1 as transmission units, respectively. Figure 10(b) shows its pseudo-isomorphic graph. From Tables 1 and 2, we have  $G_{4,1} = G_{\mathcal{I}})_{IU-1} = 1$ ,  $G_{4,2} = G_{\mathcal{I}})_{TU-6} = e_{54} (1 - e_{16} e_{65})$  and  $G_{4,3} = G_{\mathcal{I}})_{TU-1} = e_{37}$ , the input angular displacement,  $q_{4,0}$ , can be rewritten from Equation (24) as

$$\begin{aligned} q_{4,0} &= \sum_{m=1}^3 \left[ \left( \prod_{n=1}^m G_{4,n} \right) \theta_m \right] = G_{4,1} \theta_1 + G_{4,1} G_{4,2} \theta_2 + G_{4,1} G_{4,2} G_{4,3} \theta_3 \\ &= \theta_1 + e_{54} (1 - e_{16} e_{65}) \theta_2 + e_{37} e_{54} (1 - e_{16} e_{65}) \theta_3 \end{aligned} \quad (25)$$

For the mechanical transmission line corresponding to Fig. 7(a), IU-1 is used as input unit and TU-1 is used as the first and the last transmission units. From Tables 1 and 2, we have  $G_{4,1} = G_{\mathcal{I}})_{IU-1} = 1$ ,  $G_{4,2} = G_{\mathcal{I}})_{TU-1} = e_{54}$  and  $G_{4,3} = G_{\mathcal{I}})_{TU-1} = e_{35}$ . Thus, Equation (24) can be rewritten for the input angular displacement,  $q_{4,0}$ ,

$$q_{4,0} = \sum_{m=1}^3 \left[ \left( \prod_{n=1}^m G_{4,n} \right) \theta_m \right] = \theta_1 + e_{54} \theta_2 + e_{54} e_{35} \theta_3 \quad (26)$$

For the mechanical transmission line corresponding to Fig. 7(b), from Tables 1 and 2, we have  $G_{6,1} = G_{\mathcal{I}})_{IU-1} = 1$ ,  $G_{6,2} = G_{\mathcal{I}})_{TU-1} = e_{76}$  and  $G_{6,3} = G_{\mathcal{I}})_{TU-1} = e_{37}$ . Thus, the input angular displacement,  $q_{6,0}$ , from Equation (24), can be written as

$$q_{6,0} = \sum_{m=1}^3 \left[ \left( \prod_{n=1}^m G_{6,n} \right) \theta_m \right] = \theta_1 + e_{76} \theta_2 + e_{76} e_{37} \theta_3 \quad (27)$$

For the direct drive mechanical transmission line corresponding to Fig. 6(a), we have  $G_{1,1} = G_{\mathcal{I}})_{IU-1} = 1$  and

$$q_{1,0} = G_{1,1} \theta_1 = \theta_1 \quad (28)$$

Let  $\Phi = [q_{1,0}, q_{4,0}, q_{6,0}]^T$  be the displacement vector associated with the actuator-space and  $\Theta = [\theta_1, \theta_2, \theta_3]^T$  be the displacement vector associated with the joint-space. Then displacement vectors at actuator-space and joint-space can be derived by Equations (26)–(28) and be written as

$$\Phi = A^T \Theta \quad (29)$$

where  $()^T$  is the transpose of  $()$  and  $A$  is the structure matrix [7] written as

$$A = \begin{bmatrix} G_{1,1} & G_{4,1} & G_{6,1} \\ 0 & G_{4,1} G_{4,2} & G_{6,1} G_{6,2} \\ 0 & G_{4,1} G_{4,2} G_{4,3} & G_{6,1} G_{6,2} G_{6,3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & e_{54} & e_{76} \\ 0 & e_{54} e_{35} & e_{76} e_{37} \end{bmatrix} \quad (30)$$

It can also be shown that the equation relating the joint torques to the input torques is given by [7]

$$\tau = A \xi \quad (31)$$

where  $\tau = [\tau_1, \tau_2, \tau_3]^T$  denotes the resultant joint torques about joint axes  $Z_0, Z_1,$  and  $Z_2,$  and  $\xi = [\xi_1, \xi_4, \xi_6]^T$  denotes the input torques applied at links 1, 4, and 6, respectively.

From Equations (30) and (31), it can be seen that the  $i$ -th row of the structure matrix  $A$  describes how the resultant torque at joint " $i$ " is affected by the input actuators. On the other hand, the  $k$ -th column of matrix  $A$  describes how the torque of an input actuator " $k$ " is transmitted to various joints of a mechanism. Thus, the  $(i, k)$  element of structure matrix  $A$  represents how the torque of input actuator " $k$ " is transmitted to joint " $i$ " of the mechanism. Note that the elements of structure matrix  $A$  are function of the forward and backward gains of the input and transmission units of the associated mechanical transmission lines. Hence, the forward and backward gains of the input unit and transmission units provide a better insight of the effects on the torque transmission of each mechanical transmission line.

### SUMMARY

A systematic methodology for the derivation of kinematic analysis for a general class of geared robot manipulators has been developed. The approach is based on the concept that a geared robot manipulator can be viewed as the integration of an equivalent open-loop chain and several mechanical transmission lines and each mechanical transmission line as an input unit and several transmission units connected in series. It is shown that forward gains and/or backward gains of admissible input and transmission units can be systematically derived. It is also shown that the angular displacement of the connecting links with respect to its associated primary links can be systematically represented as a function of joint angular displacements and forward and/or backward gains by a unit-by-unit evaluation procedure. This method leads to an automated derivation of the kinematic relations between the actuator-space and joint-space without the need of solving a set of linear equations simultaneously. This unit-by-unit evaluation procedure provides a better insight of the effects of the input and transmission units on the torque transmission of each mechanical transmission line.

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