**Lung-Wen Tsai**
Professor.
Mechanical Engineering Department and Institute for Systems Research,
The University of Maryland,
College Park, MD 20742

**Dar-Zen Chen**
Associate Professor.

**Ta-Wei Lin**
Research Assistant.

Department of Mechanical Engineering,
National Taiwan University,
Taipei, Taiwan, 106, ROC

# Dynamic Analysis of Geared Robotic Mechanisms Using Graph Theory

*In this paper, an efficient methodology for the dynamic analysis of geared robotic mechanisms is presented. The approach makes use of the topological structure of mechanical transmission lines to simplify the analysis. The links in a geared robotic mechanism are divided into several levels. Each level contains one primary link and several secondary links. It is shown that the inverse dynamic problem can be solved from the highest level links toward the base link, one or two levels at a time depending on the topological structure of the mechanical transmission lines. Thus, the inverse dynamic problem can be solved without the need of simultaneously solving the entire system of equations. A two degree-of-freedom manipulator is used to illustrate the procedure.*

## Introduction

Many general purpose computer programs such as ADAMS (MDI, 1981), DADS (CADSI, 1988), and SD/FAST (SDI, 1988) are capable of modeling the kinematics and dynamics of mechanical systems. However, they generally have limited capability in modeling systems with higher-pair joints. That is, they often require user supplied subroutines to model more complicated mechanical systems such as planetary gear trains with floating carriers. Further, they are usually not efficient in computation time. For this reason, there has been an increasing interest in the development of efficient methodologies specifically tailored for the dynamic analysis of robot manipulators (Hollerbach, 1980; Armstrong, 1979; Orin et al., 1979; Luh et al., 1980; Walker and Orin, 1982; and Lee et al., 1983).

Although most robot manipulators are driven by motors through gear trains, most of the previous studies did not address the effects of gear trains. Strictly speaking, these formulations are only applicable to direct-drive manipulators. This paper presents an efficient methodology for the dynamical analysis of geared robotic mechanisms. In what follows, we first review the canonical graph representation of geared robotic mechanisms. Then, we present the kinematic analysis of the links. Finally, we show how the inverse dynamics can be solved level-by-level using the proposed approach.

## Graph Representation

Figure 1 shows a two degree-of-freedom (dof) planar manipulator with two independent rotations of the links about axes $\mathbf{z}_0$ and $\mathbf{z}_1$. Links 3 and 5 serve as the input links while link 2 is the output link. Torque exerted by motor 1 is transmitted to link 1 via the gear pair mounted on links 1 and 5; and torque exerted by motor 2 is transmitted to link 2 by the gear pairs mounted on links 3 and 4, and 4 and 2, respectively.

In a graph representation, links are denoted by vertices, turning pairs by thin edges, and gear pairs by heavy edges. The thin edges are labeled according to their axes locations in space. Figure 2($a$) shows the graph representation for the two-dof manipulator. By rearranging coaxial revolute joints, a rooted graph called canonical graph (Tsai, 1988) can be obtained such that all edges lying on a thin-edged path traced from the root to any other vertex have different edge labels. Further, the vertices can be divided into several levels (Chatterjee and Tsai, 1994). The ground-level vertex represents the root, the first-level vertices are one thin edge away from the root, the second-level vertices are two thin edges away from the root, etc. Figure 2($b$) shows the canonical graph of the two-dof manipulator. In the canonical graph, the kinematic chain represented by the thin-edged path which starts from the base link and ends at the output link is defined as the equivalent open-loop chain (EOLC) (Tsai, 1988). Each link in the EOLC is referred to as a primary link, and all other links are called the secondary links. A secondary link $j$ is said to be carried by a primary link $i$ if it is connected to link $i$ by one thin edge. As shown in Fig. 2($b$), links 0, 1 and 2 are the primary links, links 3, 4, and 5 are the secondary links. Secondary links 3 and 5 are carried by the primary link 0 and link 4 is carried by the primary link 1.

## Kinematics of Primary Links

Beginning from the base link, primary links are numbered sequentially from 0 to $n$. To facilitate the analysis, a coordinate system ($\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$) is attached to the distal end of link $i$ in accordance with the D-H Convention (Denavit and Hartenberg, 1955). Let $\theta_{i,i-1}$ be the joint angle between $\mathbf{x}_{i-1}$ axis to $\mathbf{x}_i$ axis, $a_{i,i-1}$ and $\alpha_{i,i-1}$ be offset distance and twist angle between $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$ axes, and $d_{i,i-1}$ be translational distance measured from $\mathbf{x}_{i-1}$ axis to $\mathbf{x}_i$ axis along $\mathbf{z}_{i-1}$ axis. The transformation between the $i$-th and ($i-1$)-th coordinate systems can be described by a matrix ${}^{i-1}\mathbf{H}_i$

$$ {}^{i-1}\mathbf{H}_i = \begin{bmatrix} C\theta_{i,i-1} & -C\alpha_{i,i-1}S\theta_{i,i-1} & S\alpha_{i,i-1}S\theta_{i,i-1} & a_{i,i-1}C\theta_{i,i-1} \\ S\theta_{i,i-1} & C\alpha_{i,i-1}C\theta_{i,i-1} & -S\alpha_{i,i-1}C\theta_{i,i-1} & a_{i,i-1}S\theta_{i,i-1} \\ 0 & C\alpha_{i,i-1} & C\alpha_{i,i-1} & d_{i,i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

where $C\theta_{i,i-1} = \cos(\theta_{i,i-1})$, $S\theta_{i,i-1} = \sin(\theta_{i,i-1})$, $C\alpha_{i,i-1} = \cos(\alpha_{i,i-1})$, and $S\alpha_{i,i-1} = \sin(\alpha_{i,i-1})$.

Since the primary links form an open-loop chain, their velocities and accelerations can be computed by using the recursive method presented by Walker and Orin (1982). The computation starts from the first moving link toward the end-effector link. The recursive equations can be summarized as follows:

$$ {}^{i}\boldsymbol{\omega}_i = {}^{i}\mathbf{R}_{i-1}({}^{i-1}\boldsymbol{\omega}_{i-1} + \dot{\theta}_{i,i-1}\,{}^{i-1}\mathbf{z}_{i-1}) \quad (2)$$

$$ {}^{i}\dot{\boldsymbol{\omega}}_i = {}^{i}\mathbf{R}_{i-1}({}^{i-1}\dot{\boldsymbol{\omega}}_{i-1} + \ddot{\theta}_{i,i-1}\,{}^{i-1}\mathbf{z}_{i-1} + {}^{i-1}\boldsymbol{\omega}_{i-1} \times \dot{\theta}_{i,i-1}\,{}^{i-1}\mathbf{z}_{i-1}) \quad (3)$$

Contributed by the Mechanisms Committee for publication in the JOURNAL OF MECHANICAL DESIGN. Manuscript received Jan. 1996. Associate Technical Editor: K. Kazerounian.

  

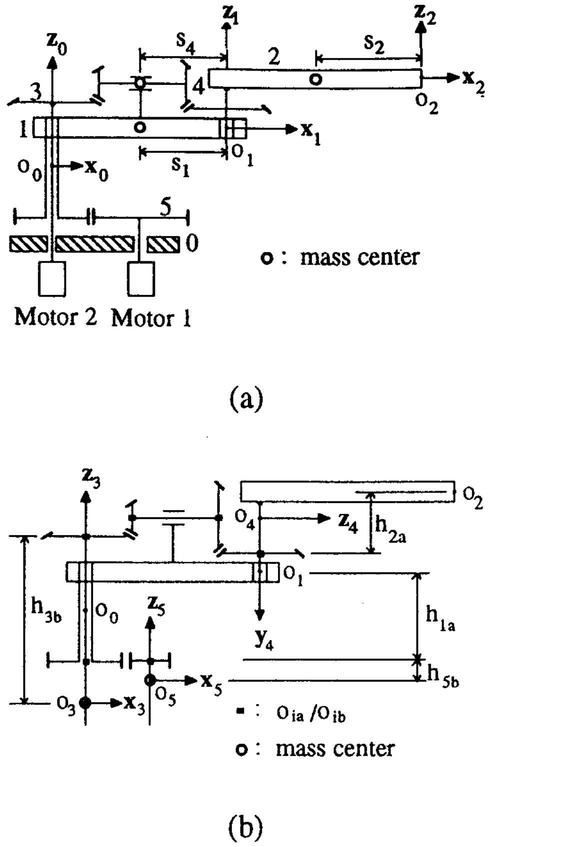


Fig. 1 A two-dof geared robot manipulator (*a*) coordinate systems of primary links (*b*) coordinate systems of secondary links

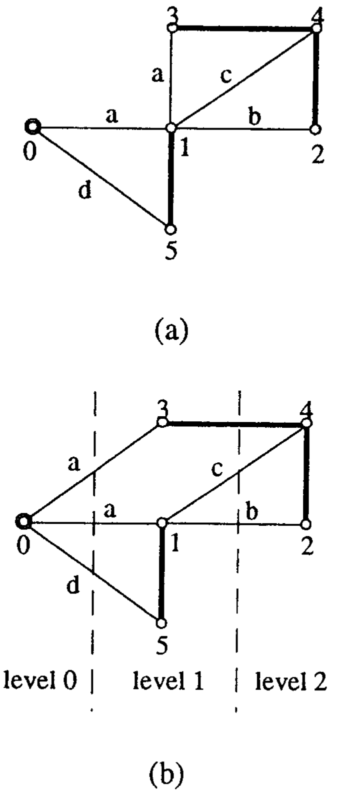


Fig. 2 (*a*) Graph representation (*b*) canonical graph

$$ {}^i\dot{\mathbf{v}}_i = {}^i\dot{\boldsymbol{\omega}}_i \times {}^i\mathbf{p}_{i,i-1} + {}^i\mathbf{R}_{i-1}{}^{i-1}\dot{\mathbf{v}}_{i-1} + {}^i\boldsymbol{\omega}_i \times ({}^i\boldsymbol{\omega}_i \times {}^i\mathbf{p}_{i,i-1}) \quad (4)$$

$$ {}^i\dot{\mathbf{v}}_{ic} = {}^i\dot{\boldsymbol{\omega}}_i \times {}^i\mathbf{r}_{ic} + {}^i\dot{\mathbf{v}}_i + {}^i\boldsymbol{\omega}_i \times ({}^i\boldsymbol{\omega}_i \times {}^i\mathbf{r}_{ic}) \quad (5)$$

where $\boldsymbol{\omega}_i$ is angular velocity vector of link $i$, $\mathbf{v}_i$ is velocity vector of the origin $o_i$, $\mathbf{v}_{ic}$ is velocity of the center of mass of link $i$, $o_{ic}$, $\mathbf{r}_{ic}$ is position vector of link $i$ defined from $o_i$ to $o_{ic}$, ${}^{i-1}\mathbf{z}_{i-1} = [0, 0, 1]^T$ is an unit vector defined along $\mathbf{z}_{i-1}$ axis, ${}^i\mathbf{R}_{i-1}$ is the transpose of the upper left 3 by 3 sub-matrix of ${}^{i-1}\mathbf{H}_i$, and ${}^i\mathbf{p}_{i,i-1}$, can be written as

$$ {}^i\mathbf{p}_{i,i-1} = [a_{i,i-1},\, d_{i,i-1}S\alpha_{i,i-1},\, d_{i,i-1}C\alpha_{i,i-1}]^T \quad (6)$$

## Kinematics of Secondary Links

A coordinate system $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ is defined for each secondary link according to the D-H Convention. For the case of a secondary link $j$ which is carried by a primary link $i$, the transformation between the $i$-th and $j$-th coordinate systems can be described by a matrix ${}^iB_j$

$$ {}^i\mathbf{B}_j = \begin{bmatrix} C\theta_{j,i} & -C\alpha_{j,i}S\theta_{j,i} & S\alpha_{j,i}S\theta_{j,i} & a_{j,i}C\theta_{j,i} \\ S\theta_{j,i} & C\alpha_{j,i}C\theta_{j,i} & -S\alpha_{j,i}C\theta_{j,i} & a_{j,i}S\theta_{j,i} \\ 0 & C\alpha_{j,i} & C\alpha_{j,i} & d_{j,i} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7)$$

Let links $j$ and $k$ be a gear pair and $i$ be the carrier. Then, links $i$, $j$ and $k$ form a fundamental circuit and a fundamental circuit equation can be written as:

$$ \boldsymbol{\theta}_{j,i} = e_{kj}\boldsymbol{\theta}_{k,i} \quad (8)$$

where $e_{kj}$ denotes gear ratio for the gear pair mounted on links $k$ and $j$, positive or negative according as a positive rotation of gear $k$ results in a positive or negative rotation of gear $j$ about their pre-defined axes of rotation.

Let $i$, $j$ and $k$ be coaxial links, then the relative motion among these links are related by the following coaxial condition:

## Nomenclature

In this paper, all vector differentiations are taken with respect to the base frame. A vector with no leading superscript indicates that the vector is expressed in the base frame or any reference frame, and a vector with a leading superscript identical to its subscript implies that the vector is expressed in its own link frame.

$f_{i,j}$ = force acting on link $i$ by link $j$ at joint $[i, j]$.

$\mathbf{n}_{i,j}$ = moment acting on link $i$ by link $j$ at joint $[i, j]$.

$\mathbf{p}_{j,i}$ = position vector defined from $o_i$ to $o_j$.

$\mathbf{r}_{j,i}$ = position vector defined from $o_i$ to joint $[i, j]$.

$\mathbf{r}_{ja}$ = position vector of link $j$ defined from $o_j$ to the mesh point of the input gear.

$\mathbf{r}_{jb}$ = position vector of a secondary link $j$ defined from $o_j$ to the mesh point of the output gear.

$\theta_{j,i}$ = angular displacement of link $j$ with respect to link $i$.

$\lambda_{ia}$ ($\lambda_{ib}$) = radius of the input (output) gear mounted on link $i$.

$$\theta_{i,k} = \theta_{i,j} + \theta_{j,k} \tag{9}$$

Assuming that all the secondary links are symmetric about their axes of rotation, their centers of mass can be considered as fixed points in their corresponding carriers. Hence, the acceleration and angular acceleration of the center of mass of a secondary link $j$ can be computed by modifying Eq. (3) and Eq. (5) as

$$^j\dot{\boldsymbol{\omega}}_j = {}^j\mathbf{T}_i{}^i\dot{\boldsymbol{\omega}}_i + \ddot{\theta}_{j,i}{}^j\mathbf{z}_j + {}^j\mathbf{T}_i{}^i\boldsymbol{\omega}_i \times \dot{\theta}_{j,i}{}^j\mathbf{z}_j \tag{10}$$

$$^j\dot{\mathbf{v}}_{jc} = {}^j\mathbf{T}_i{}^i\dot{\boldsymbol{\omega}}_i \times ({}^j\mathbf{p}_{j,i} + {}^j\mathbf{r}_{jc}) + {}^j\mathbf{T}_i{}^i\dot{\mathbf{v}}_i + {}^j\mathbf{T}_i{}^i\boldsymbol{\omega}_i \times [{}^j\mathbf{T}_i{}^i\boldsymbol{\omega}_i \times ({}^j\mathbf{p}_{j,i} + {}^j\mathbf{r}_{jc})] \tag{11}$$

where $^j\mathbf{T}_i$ is the transpose of the upper left 3 by 3 sub-matrix of $^i\mathbf{B}_j$ and $^j\mathbf{p}_{j,i}$ can be written as

$$^j\mathbf{p}_{j,i} = [a_{j,i}, d_{j,i}S\alpha_{j,i}, d_{j,i}C\alpha_{j,i}]^T \tag{12}$$

## Gear Contact Forces

A heavy-edged path which begins from an input link and ends at a primary link, is called a mechanical transmission line (Chang and Tsai, 1990). The first secondary link in a mechanical transmission line contains only one output gear since the other end is attached to a motor drive shaft. All the other secondary links contain two gears, one input gear and one output gear. Sometimes, the two gears on a secondary link can be combined into a single gear, but it still meshes with two other gears in a mechanical transmission line. For the manipulator shown in Fig. 1, there are two mechanical transmission lines: $\langle 5 - 1 \rangle$ and $\langle 3 - 4 - 2 \rangle$. The input gear on the secondary link 4 meshes with the gear mounted on link 3 and the output gear meshes with the gear mounted on link 2.

Figure 3 shows a typical secondary link $j$ carried by a primary link $i$. Assume that each of the input and output gears meshes with a gear mounted on the secondary links $j - 1$ and $j + 1$, respectively. Also, let a primary link $k$ be the carrier of the gear pair $[j - 1, j]$ and a primary link $k + 1$ be the carrier of the gear pair $[j, j + 1]$. Note that the input and output gears on a secondary link may have two distinct carriers. Typically, a primary link which carries a secondary link $j$ is the carrier of the gear pair $[j - 1, j]$ or gear pair $[j, j + 1]$. Let $\mathbf{z}_j$ be an unit vector defined along the positive axis of rotation of a secondary link $j$, two local coordinate systems can be defined as follows. (1) The input local coordinate system $(\mathbf{x}_{ja}, \mathbf{y}_{ja}, \mathbf{z}_{ja})$: where $\mathbf{x}_{ja}$ is an unit vector pointing from the center of the input gear, $o_{ja}$, to the meshing point of the gear pair $[j - 1, j]$, $\mathbf{z}_{ja} = \mathbf{z}_j$, and $\mathbf{y}_{ja}$ is determined from the right-hand rule. (2) The output local coordinate system $(\mathbf{x}_{jb}, \mathbf{y}_{jb}, \mathbf{z}_{jb})$: where $\mathbf{x}_{jb}$ is an unit vector pointing from the center of the output gear, $o_{jb}$, to the meshing point of the gear pair $[j, j + 1]$, $\mathbf{z}_{jb} = \mathbf{z}_j$, and $\mathbf{y}_{jb}$ is determined from the right-hand rule. Since $(\mathbf{x}_{ja}, \mathbf{y}_{ja}, \mathbf{z}_{ja})$ is fixed in the coordinate system of primary link $k$, $(\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k)$, the transformation between them can be described as

$$^k\mathbf{Q}_{ja} = \begin{bmatrix} \mathbf{x}_k \cdot \mathbf{x}_{ja} & \mathbf{x}_k \cdot \mathbf{y}_{ja} & \mathbf{x}_k \cdot \mathbf{z}_{ja} \\ \mathbf{y}_k \cdot \mathbf{x}_{ja} & \mathbf{y}_k \cdot \mathbf{y}_{ja} & \mathbf{y}_k \cdot \mathbf{z}_{ja} \\ \mathbf{z}_k \cdot \mathbf{x}_{ja} & \mathbf{z}_k \cdot \mathbf{y}_{ja} & \mathbf{z}_k \cdot \mathbf{z}_{ja} \end{bmatrix} \tag{13}$$

Similarly, the transformation between coordinate systems $(\mathbf{x}_{jb}, \mathbf{y}_{jb}, \mathbf{z}_{jb})$ and $(\mathbf{x}_{k+1}, \mathbf{y}_{k+1}, \mathbf{z}_{k+1})$ can also be defined. Note that if secondary link $j$ is an input link, only $(\mathbf{x}_{jb}, \mathbf{y}_{jb}, \mathbf{z}_{jb})$ can be defined. Neglecting frictional force in a gear mesh, the contact force exerted on link $j$ by link $j + 1$, $\mathbf{g}_{j,j+1}$, can be expressed as

$$^{jb}\mathbf{g}_{j,j+1} = [-T\phi_{jb}C\psi_{jb}|t_{j,j+1}|, t_{j,j+1}, \delta_{jb}T\phi_{jb}S\psi_{jb}|t_{j,j+1}|]^T \tag{14}$$

where $t_{j,j+1}$ is the tangential component of the contact force, $|()|$ is the absolute value of ( ), $\phi_{jb}$ is the pressure angle, $\psi_{jb}$ is the pitch cone angle, $T\phi_{jb} = \tan(\phi_{jb})$, and $\delta_{jb} = 1$ if the cone apex of the output gear on link $j$ falls on the negative side of $\mathbf{z}_j$ axis, otherwise $\delta_{jb} = -1$.

From Eq. (14), it can be seen that the $\mathbf{x}_{jb}$ component of $\mathbf{g}_{j,j+1}$ is always negative, the $\mathbf{z}_{jb}$ component of $\mathbf{g}_{j,j+1}$ depends on the apex location. Hence, only one unknown parameter, $t_{j,j+1}$, is necessary for the specification of a gear contact force.

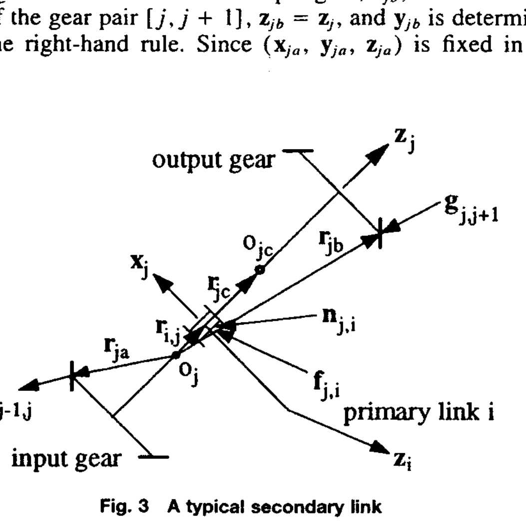


Fig. 3 A typical secondary link

## Dynamic Analysis

In the absence of friction, modeling the reaction force in a revolute joint requires five unknowns: three forces and two torques since a revolute joint cannot transmit torque about its axis of rotation. Hence, each heavy edge in the graph of a mechanism contains one unknown and each thin edge contains five unknowns.

The force and moment balance equations of an individual link are the Newton and Euler's equation. Theoretically, it is possible to establish six balance equations for each link and solve for all the unknowns simultaneously. However, this would be computationally intensive and inefficient. If there are six or fewer unknowns in the balance equations of a single link, then these equations can be solved independently. Once these equations are solved, the balance equations for the other link may become solvable. This procedure can continue throughout the system until all the unknowns are solved. It turns out that this is the case for geared robotic mechanisms with a pseudo-triangular structure (Chang and Tsai, 1990). A geared robotic mechanism is said to have a pseudo-triangular structure if every primary link is driven by a mechanical transmission line. The procedure starts from a primary link of the highest level followed by the secondary link of the same level. Then, the process repeats itself for next level until all the unknown joint forces and moments are found. The procedure for mechanisms with a non-pseudo-triangular structure is similar, except that the balance equations need to be solved simultaneously for $n$ levels of primary links if there are $n$ mechanical transmission lines connected to a primary link. Once the joint forces associated with these $n$ primary links are found, reaction forces associated with the secondary links can be solved recursively from the higher level vertices to the lower level vertices. Since there are only a few linear equations to be solved simultaneously, this link-by-link recursive procedure can be computationally more efficient than the general purpose computer programs.

**Dynamics of Primary Links.** Figure 4 shows the free body diagram of a typical primary link $i$ with two revolute joints connecting it to primary links $i - 1$ and $i + 1$. Assume that primary link $i$ carries $j$ secondary links and is the last link of $k$ mechanical transmission lines. The force and moment balance equations for link $i$ can be written in the following recursive forms:

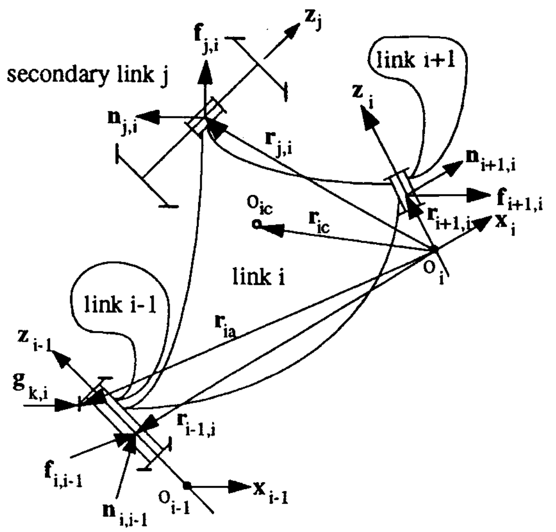


Fig. 4 A typical primary link

$$\mathbf{f}_{i,i-1} - \sum_{k} \mathbf{g}_{k,i} = \mathbf{F}_i^* + \mathbf{f}_{i+1,i} + \sum_{j} \mathbf{f}_{j,i} \quad (15)$$

and

$$\mathbf{n}_{i,i-1} - (\mathbf{r}_{ic} - \mathbf{r}_{i-1,i}) \times \mathbf{f}_{i,i-1} - \sum_{k} [(\mathbf{r}_{ia} - \mathbf{r}_{ic}) \times \mathbf{g}_{k,i}]$$

$$= \mathbf{N}_i^* + \mathbf{n}_{i+1,i} - (\mathbf{r}_{ic} - \mathbf{r}_{i+1,i}) \times \mathbf{f}_{i+1,i} + \sum_{j} \mathbf{n}_{j,i} + \sum_{j} [(\mathbf{r}_{j,i} - \mathbf{r}_{ic}) \times \mathbf{f}_{j,i}] \quad (16)$$

where $\mathbf{F}_i^*$ and $\mathbf{N}_i^*$ are the inertia force and moment vectors of primary link $i$ and can be computed as follows.

$$\mathbf{F}_i^* = m_i \dot{\mathbf{v}}_{ic} \quad (17)$$

and

$$\mathbf{N}_i^* = {}^c\mathbf{I}_i \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times ({}^c\mathbf{I}_i \boldsymbol{\omega}_i) \quad (18)$$

where $m_i$ is mass of link $i$, and ${}^c\mathbf{I}_i$ is the inertia tensor of link $i$ with respect to a center of mass coordinate system which has the same orientation as the $i$-th coordinate system.

The vectors in the right-hand sides of Eqs. (15) and (16), $\mathbf{f}_{i+1,i}$, $\mathbf{n}_{i+1,i}$, $\mathbf{f}_{j,i}$ and $\mathbf{n}_{j,i}$, are computed from the balance equations of the previous primary link. For the end-effector, these four vectors represent the end-effector output force and moment. The left-hand sides of Eqs. (15) and (16) contain all the unknown force and moment vectors: $\mathbf{f}_{i,i-1}$, $\mathbf{n}_{i,i-1}$ and $\mathbf{g}_{k,i}$. For robotic mechanisms with pseudo-triangular structure, $\mathbf{f}_{i,i-1}$, $\mathbf{n}_{i,i-1}$ and $\mathbf{g}_{k,i}$ constitute six scalar unknowns. Hence, they can be solved one link at a time.

**Dynamics of the Secondary Links.** Referring to Fig. 3, the force and moment balance equations for a secondary link $j$ can be written as:

$$\mathbf{f}_{j,i} - \mathbf{g}_{j-1,j} = \mathbf{F}_j^* - \mathbf{g}_{j,j+1} \quad (19)$$

and

$$\mathbf{n}_{j,i} - (\mathbf{r}_{ja} - \mathbf{r}_{jc}) \times \mathbf{g}_{j-1,j} + (\mathbf{r}_{i,j} - \mathbf{r}_{jc}) \times \mathbf{f}_{j,i} = \mathbf{N}_j^* - (\mathbf{r}_{jb} - \mathbf{r}_{jc}) \times \mathbf{g}_{j,j+1} \quad (20)$$

where $\mathbf{F}_j^*$ and $\mathbf{N}_j^*$ are the inertia force and moment for link $j$ and can be computed similarly by Eqs. (17–18).

The vector $\mathbf{g}_{j,j+1}$ in the right-hand sides of Eqs. (19) and (20) are computed from the balance equations of the higher level links. The left-hand sides of Eqs. (19) and (20) contain all the unknown force and moment vectors: $\mathbf{f}_{j,i}$, $\mathbf{n}_{j,i}$ and $\mathbf{g}_{j-1,j}$. Once the joint forces and moments of the higher level links and the primary link of the same level are solved, the unknown forces and moments of secondary links in Eqs. (19) and (20) can be solved one at a time. For the secondary link, any vector defined in its input and/or output local coordinate system can be readily transformed into the coordinate systems of their respective carrier by using Eq. (13).

The reaction forces and moments solved from the above procedure are the forces and moments acting on the joints of the canonical representation of a mechanism. These forces and moments should be converted into the force and moments acting on the original mechanism. This can be accomplished by solving a set of equivalent force systems associated with the coaxial links.

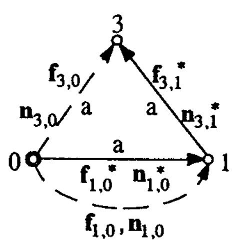

Fig. 5 Equivalent force system among coaxial links

## Example

For the manipulator shown in Fig. 1($a$) and ($b$), from Eqs. (7) and (12), we have ${}^1\mathbf{p}_{1,0} = [a_1, 0, d_1]^T$, ${}^2\mathbf{p}_{2,1} = [a_2, 0, d_2]^T$, ${}^3\mathbf{p}_{3,0} = [0, 0, -d_{30}]^T$, ${}^5\mathbf{p}_{5,0} = [a_{50}, 0, -d_{50}]^T$, and ${}^4\mathbf{p}_{4,1} = [0, -d_{41}, 0]^T$. Let ${}^1\mathbf{r}_{1c} = [-s_1, 0, 0]^T$, ${}^2\mathbf{r}_{2c} = [-s_2, 0, 0]^T$, ${}^3\mathbf{r}_{3c} = [0, 0, 0]^T$, ${}^4\mathbf{r}_{4c} = [0, 0, -s_4]^T$, and ${}^5\mathbf{r}_{5c} = [0, 0, 0]^T$, then

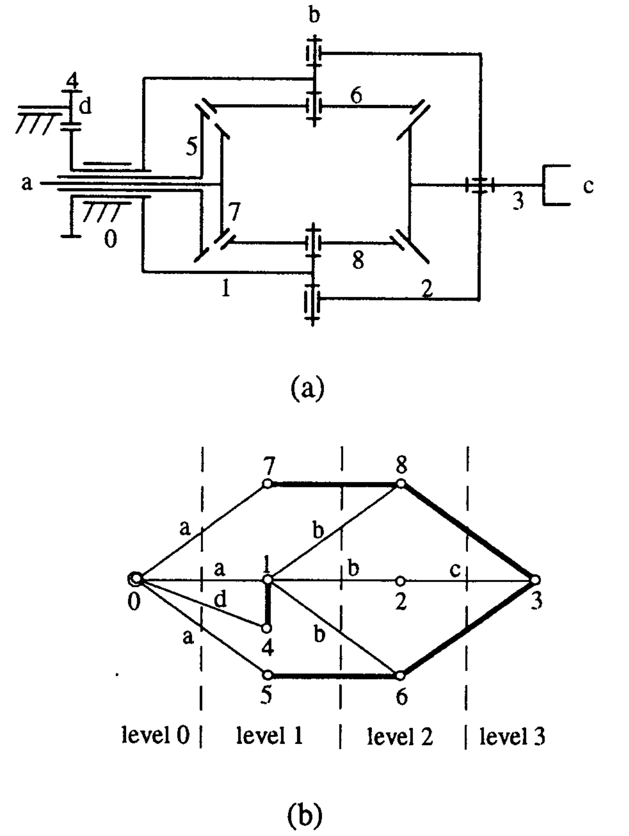


Fig. 6 ($a$) A Bendix wrist ($b$) canonical graph the mass center accelerations and angular accelerations of the primary links 1, 2 and secondary links 3, 4, and 5 can be computed from Eqs. (2-5) and Eqs. (10) and (11), respectively. The position vectors required for the evaluation of Eqs. (15) and (16) are: ${}^0\mathbf{r}_{1,0} = [0, 0, 0]^T$, ${}^1\mathbf{r}_{2,1} = [0, 0, 0]^T$, ${}^2\mathbf{r}_{3,2} = [0, 0, 0]^T$, ${}^0\mathbf{r}_{3,0} = [0, 0, -d_{30}]^T$, ${}^1\mathbf{r}_{4,1} = [-s_4, 0, d_{41}]^T$, ${}^0\mathbf{r}_{5,0} = [a_{50}, 0, -d_{50}]^T$, ${}^1\mathbf{r}_{1a} = [-a_1 + \lambda_{1a}, 0, -h_{1a}]^T$, and ${}^2\mathbf{r}_{2a} = [-a_2 - \lambda_{1a}, 0, -h_{2a}]^T$. Position vectors required for the evaluation of Eqs. (19) and (20) are: ${}^3\mathbf{r}_{3b} = [\lambda_{3b}, 0, h_{3b}]^T$, ${}^4\mathbf{r}_{4a} = [0, \lambda_{4a}, -a_1 + \lambda_{3a}]^T$, ${}^4\mathbf{r}_{4b} = [0, r_{4b}, -\lambda_{2a}]^T$, and ${}^5\mathbf{r}_{5b} = [-r_{5b}, 0, h_{5b}]^T$. Note that $\mathbf{r}_{1,4} = -\mathbf{p}_{4,1} + \mathbf{r}_{4,1}$, $\mathbf{r}_{0,3} = -\mathbf{p}_{3,0} + \mathbf{r}_{3,0}$, and $\mathbf{r}_{0,5} = -\mathbf{p}_{5,0} + \mathbf{r}_{5,0}$.

From fundamental circuit equations and coaxial condition, angular displacements of the secondary links can be expressed in terms of the joint displacements as

$$\theta_{4,1} = e_{24}\theta_{2,1} \quad (21)$$

and

$$\begin{bmatrix} \theta_{3,0} \\ \theta_{5,0} \end{bmatrix} = \begin{bmatrix} 1 & e_{24}e_{43} \\ e_{15} & 0 \end{bmatrix} \begin{bmatrix} \theta_{1,0} \\ \theta_{2,1} \end{bmatrix} = \mathbf{A}^T \begin{bmatrix} \theta_{1,0} \\ \theta_{2,1} \end{bmatrix} \quad (22)$$

where **A** is called the structure matrix (Chang and Tsai, 1990).

The dynamic loads on the bearings and gear teeth of the mechanism can be evaluated in the following sequence: $\{2 \rightarrow 4, 1 \rightarrow 3$ and $5\}$. Figure 5 shows the equivalent force system among coaxial links, 0, 1, and 3. In Fig. 5, solid lines represent actual reaction forces while dashed lines represent canonical reaction forces. The actual reaction forces, $\mathbf{f}^*_{3,1}$ and $\mathbf{n}^*_{3,1}$, can be determined from the following balance equations about the mass center of link 3:

$$\mathbf{f}^*_{3,1} = \mathbf{f}_{3,0} \quad (23)$$

and

$$\mathbf{n}^*_{3,1} + (\mathbf{r}_{1,3} - \mathbf{r}_{3c}) \times \mathbf{f}^*_{3,1} = \mathbf{n}_{3,0} + (\mathbf{r}_{0,3} - \mathbf{r}_{3c}) \times \mathbf{f}_{3,0} \quad (24)$$

Similarly, the actual reaction forces, $\mathbf{f}^*_{1,0}$ and $\mathbf{n}^*_{1,0}$, can be determined from the balance equations of link 1. For the purpose of illustration, inertia tensor of link $i$ is assumed as a diagonal matrix, i.e., ${}^i\mathbf{I}_i = [I_{xx}(i), I_{yy}(i), I_{zz}(i)]^T$. Because of space limitation, the equations of motion for the two-dof manipulator are given below without derivation.

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \mathbf{M} \begin{bmatrix} \ddot{\theta}_{1,0} \\ \ddot{\theta}_{2,1} \end{bmatrix} + \mathbf{C}'_1 \begin{bmatrix} -k_1 S\theta_{2,1}\dot{\theta}^2_{1,0} \\ -k_1 S\theta_{2,1}\dot{\theta}^2_{2,1} \end{bmatrix} + \mathbf{C}'_2 \begin{bmatrix} -2k_1 S\theta_{2,1}\dot{\theta}_{1,0}\dot{\theta}_{2,1} \\ 0 \end{bmatrix} \quad (25)$$

where $\tau_1$ and $\tau_2$ are the equivalent joint torques,

$$\mathbf{M} = \begin{bmatrix} J_{10} + J_{20} + I_{zz}(3) + J_{40} + e^2_{15}I_{zz}(5) & k_2 + e_{43}e_{24}I_{zz}(3) \\ k_2 + e_{43}e_{24}I_{zz}(3) & J_{21} + e^2_{43}e^2_{24}I_{zz}(3) + e^2_{24}I_{zz}(4) \end{bmatrix} \quad (26)$$

and where

$$J_{10} = m_1(a_1 - s_1)^2 + I_{zz}(1) \quad (27a)$$

$$J_{20} = m_1a^2_1 + 2m_2a_1(a_2 - s_2)C\theta_{2,1} + m_2(a_2 - s_2)^2 + I_{zz}(2) \quad (27b)$$

$$J_{21} = m_2(a_2 - s_2)^2 + I_{zz}(2) \quad (27c)$$

$$J_{40} = m_4(a_1 - s_4)^2 + I_{yy}(4) \quad (27d)$$

$$k_1 = m_2a_1(a_2 - s_2) \quad (27e)$$

$$k_2 = k_1C\theta_{2,1} + J_{21} \quad (27f)$$

## Summary

A recursive algorithm for the dynamic analysis of a class of geared robotic mechanisms is developed. The algorithm involves two steps. In the first step, kinematic properties of primary links are computed recursively from the first moving link toward the end-effector link. Then kinematic properties of the secondary links are computed using the fundamental circuit equations and coaxial conditions. In the second step, the joint forces and moments are computed by using the Newton and Euler's equations of motion. The computation begins from the highest level primary links followed by the secondary links, one or two levels at a time working backwards toward the base. Finally, equivalent force systems are introduced for the evaluation of actual joint forces and moments among the coaxial links.

Finally, we use the Bendix wrist as shown in Fig. 6 to illustrate the need of solving two levels of dynamical equations simultaneously. In the Bendix wrist, there are two transmission lines, 5-6-3 and 7-8-3, terminated at the end-effector. The evaluation procedure is as follows. We first solve the dynamical equations of links 3 and 2 simultaneously, followed by links 6 and 8, and then links 1, 4, 5, and 7.

## References


Armstrong, W. M., 1979, "Recursive Solution to the Equations of Motion of an N-link Manipulator," *Proc. 5th World Congr., Machines and Mechanisms Theory*, Vol. 2, pp. 1343–1346.

CADSI, 1988, "DADS User's Manual," *Computer Aided Design Software, Inc.*, Rev. 5.0.

Chang, S. L., and Tsai, L. W., 1990, "Topological Synthesis of Articulated Gear Mechanisms," *IEEE Trans. on Robotics and Automation*, Vol. 6, No. 1, pp. 97–103.

Chatterjee, G., and Tsai, L. W., 1994, "Computer Aided Sketching of Epicyclic-Type Automatic Transmission Gear Trains," *Proc. of the ASME Design Technical Conf.*, DE-Vol. 71, pp. 275–282.

Denavit, J., and Hartenberg, R. S., 1955, "A Kinematic Notation for Lower Pair Mechanisms Based on Matrices," ASME *Journal of Applied Mechanics*, Vol. 77, pp. 215–221.

Hollerbach, J. M., 1980, "A Recursive Lagrangian Formulation of Manipulator Dynamics and A Comparative Study of Dynamics Formulation Complexity," *IEEE Trans., Systems, Man, Cybernetics*, Vol. SMC-10, No. 11, pp. 730–736.

Lee, C. S. G., Lee, B. H., and Nigam, R., 1983, "Development of the Generalized d'Alembert Equations of Motion for Mechanical Manipulators," *Proc. of 22nd Decision and Control Conf.*, pp. 1205–1210.

Luh, J. Y. S., Walker, M. W., and Paul, R. P., 1980, "On-line Computational Scheme for Mechanical Manipulators," ASME *Journal of Dynamic Systems, Measurement, and Control*, Vol. 120, pp. 69–76.

MDI, 1981, "User's Guide to ADAMS," *Mechanical Dynamics, Inc., 5th Ed.*, Ann Arbor, MI.

Orin, D. E., McGhee, R. B., Vukobratovic, M., and Hartoch, G., 1979, "Kinematic and Kinetic Analysis of Open-chain Linkages Utilizing Newton-Euler Methods," *Math. Biosci.*, Vol. 43, pp. 107–130.

SDI, 1988, "SD/FAST User's Manual," *Symbolic Dynamics, Inc.*

Tsai, L. W., 1988, "The Kinematics of Spatial Robotic Bevel-Gear Trains," *IEEE Journal of Robotics and Automation*, Vol. 4, No. 2, pp. 150–155.

Walker, M. W., and Orin, D. E., 1982, "Efficient Dynamic Computer Simulation of Robotic Mechanisms," ASME *Journal of Dynamic Systems, Measurement, and Control*, Vol. 104, pp. 205–211.