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Dynamic Modeling of Geared Robotic Mechanisms—The Virtual Link Approach

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Abstract

In this paper, an efficient and systematic methodology for the formulation of dynamic equations of a general class of geared robotic mechanisms is developed. The concepts of primary links and secondary links of the manipulator are used to organize the analysis. The approach is based on the idea that the contribution to the generalized inertia forces of a secondary link can be divided into two parts. The first part is due to the motion of its associated primary link and the second part is due to the motion relative to its associated primary link. The concept of virtual link is introduced to incorporate the first part of the generalized inertia force contribution with those of primary links. The second part of the generalized inertia force contribution is derived by Lagrangian formulation. It is shown that the coupling effects of secondary links in the dynamic equations of motion can be identified individually and systematically. A 3-dof geared robotic mechanism is used to illustrate the methodology. © 1998 Elsevier Science Ltd. All rights reserved.

1. Introduction

Gear trains have been used to transmit mechanical power from one rotational axis to another and to reduce rotation speeds at the output shafts for centuries. With the advent of robotic mechanisms in the work places comes a complicated form of gearing systems that provides the motions similar to those of humans. An example mechanism is the three degrees-of-freedom (dof) geared robotic mechanism (GRM) depicted in Fig. 1. In Fig. 1, links 4, 5 and 6 are the input links to which motors can be attached to drive the manipulator and link 3 is the output link, called the end–effector. Through spur or bevel gear trains, powers are transmitted to the output link 3 to which payload can be attached.

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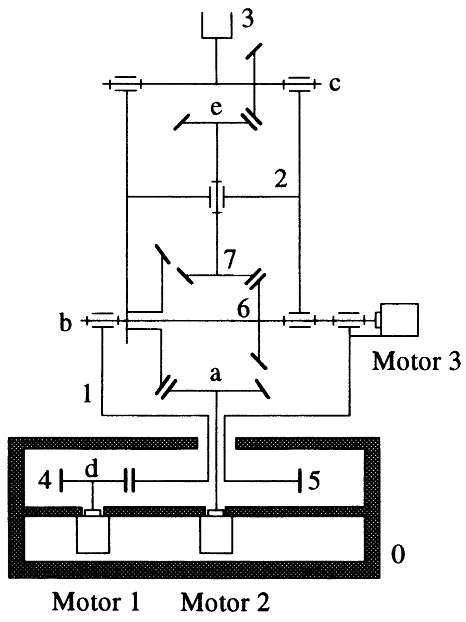

Fig. 1. A 3-dof arm.

Formulating equations of motion of a mechanical system is an important part that provides necessary information for the design of mechanical components and control laws. Although many geared robotic mechanisms are driven by motors through gear reduction units with fairly high gear ratio, most of the previous works [7, 10, 12, 14] did not elaborate on how the manipulator is driven. Strictly speaking, these formulations are only applicable to direct–drive manipulators. It is generally recognized that, for individual joint-driven manipulators, the rotational inertia of motor/rotor multiplied by the square of gear ratio should be included in the dynamic equations of motion [9, 11]. However, the coupling effect of the gear reduction units of a GRM is not addressed.

In this paper, a systematic and efficient methodology for the formulation of dynamic equations of a general class of GRM will be developed. The coupling effect of the gear reduction units of a GRM will be identified. It will be shown that the contribution to the generalized inertia force of a secondary link can be divided into two parts. The first part is due to the motion of its associated primary link, and the second part is due to its motion relative to its associated primary link. The concept of virtual link is introduced to incorporate the first part of the generalized inertia force with those of primary links. The second part of the generalized inertia force contribution of secondary links are then derived systematically and efficiently by Lagrangian formulation. This methodology provides the basis for the design of drive trains and leads to a computer automated analysis for a general class of GRMs. The three-dof manipulator shown in Fig. 1 is used as an illustrative example.

2. Structural Characteristics

In graph representation, links are denoted by vertices, turning pairs by thin edges, gear pairs by heavy edges, and thin edges are labeled according to their axes locations in space. Fig. 2(a) shows the graph representation of the example GRM. By rearranging the coaxial revolute joints, a canonical graph [13] can be uniquely defined so that there are no repeated axis labels in thin-edged paths starting from the base link. Fig. 2(b) shows its canonical graph representation. By removing all geared edges from the canonical graph, the open-loop chain starting from the base link and ending at the output link is defined as an *equivalent open-loop chain* (EOLC) [13]. Fig. 3 shows the EOLC of the example GRM. Each link in the EOLC is referred to as a *primary link*, and all other links are called *secondary links* [3]. The arrangement of secondary links describes where the input actuators are located and how the input torques are transmitted to the various joints of the manipulator. Hence, the kinematic structure of a GRM can be described by an EOLC, which is formed by primary links, and by the arrangement of secondary links which drive the joints of EOLC. In the case that secondary link j is connected with primary link i by a thin edge in the canonical graph representation, it is said that secondary link j is carried by primary link i and primary link i is associated with secondary link j. For the example GRM, links 0, 1, 2 and 3 are primary links, and links 4, 5, 6 and 7 are secondary links. Secondary links 4 and 5 are carried by primary link 0, secondary link 6 is carried by primary link 1 and secondary link 7 is carried by primary link 2.

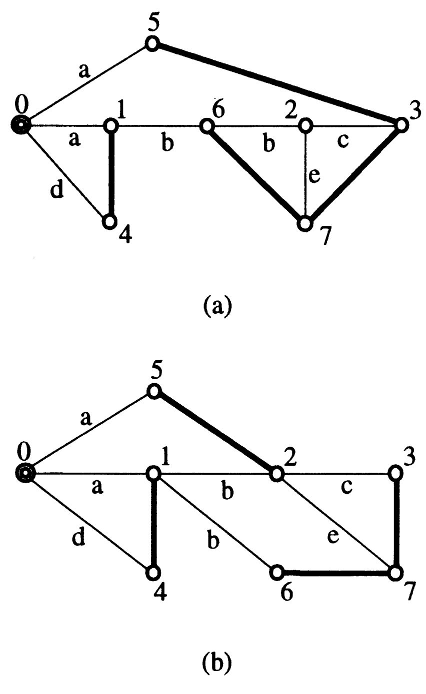

Fig. 2. (a) Graph representation. (b) Canonical graph representation.

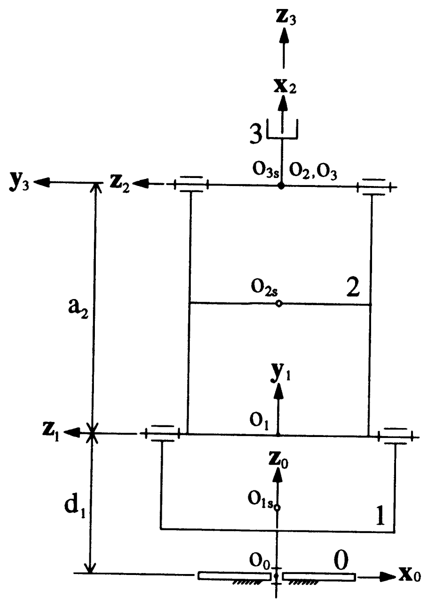

Fig. 3. The equivalent open-loop chain.

3. Kinematics of Primary Links

Beginning from the base link, primary links are numbered sequentially from 0 to n for an n-dof GRM. Based on the Denavit and Hartenberg Convention [5], a coordinate system ($\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$) is attached to primary link i and shown as in Fig. 4 where $d_{i,i-1}$ is the translational distance along the $\mathbf{z}_{i-1}$-axis, $a_{i,i-1}$ and $\alpha_{i,i-1}$ are the offset distance and twist angle between $\mathbf{z}_{i-1}$-axis and $\mathbf{z}_i$-axis, respectively. The joint angle measured from $\mathbf{x}_{i-1}$-axis to $\mathbf{x}_i$ axis along $\mathbf{z}_{i-1}$-axis, $q_{i,i-1}$, is referred to as the joint angle, θ_i. In what follows, we will concentrate on GRMs with independent joint motion, i.e., the number of joints in a GRM is equal to its dof.

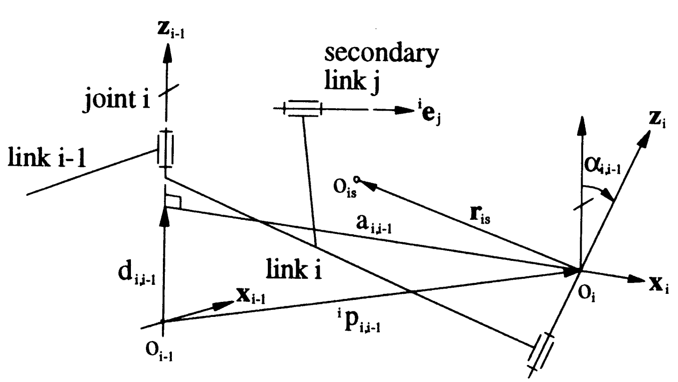

Fig. 4. Coordinate system of θ_i.

Also the joints in the EOLC are chosen as generalized coordinates. In vector notation, a leading superscript denotes the coordinate system a vector is referred to and, if it is omitted, the inertia frame is implied.

Kinematic properties of primary links can be derived by the matrix method or vector approach. One efficient approach is the forward recursive equations presented by Walker and Orin [14], which can be written as

$$^{i}\boldsymbol{\omega}_i = {}^{i}\mathbf{R}_{i-1}({}^{i-1}\boldsymbol{\omega}_{i-1} + {}^{i-1}\mathbf{z}_{i-1}\dot{\theta}_i), \tag{1}$$

$$^{i}\mathbf{v}_i = {}^{i}\boldsymbol{\omega}_i \times {}^{i}\mathbf{p}_{i,i-1} + {}^{i}\mathbf{R}_{i-1}\,{}^{i-1}\mathbf{v}_{i-1}, \tag{2}$$

$$^{i}\mathbf{v}_{is} = {}^{i}\boldsymbol{\omega}_i \times \mathbf{r}_{is} + {}^{i}\mathbf{v}_i \tag{3}$$

where ${}^{i-1}\mathbf{z}_{i-1} = [0\ 0\ 1]^{\mathrm{T}}$ is the unit vector defined along $\mathbf{z}_{i-1}$-axis, $\boldsymbol{\omega}_i$ is the angular velocity of link i, $\mathbf{v}_i$ is the linear velocity at the origin of ($\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$) coordinate system, o_i, $\mathbf{v}_{is}$ is the linear velocity at the mass center of link i, o_{is}, $\mathbf{r}_{is}$ is the position vector from o_i to o_{is}, the position vector defined from o_{i-1} to o_i, ${}^{i}\mathbf{p}_{i,i-1}$, can be written as

$$^{i}\mathbf{p}_{i,i-1} = [a_{i,i-1} \quad d_{i,i-1}S\alpha_{i,i-1} \quad d_{i,i-1}C\alpha_{i,i-1}]^{\mathrm{T}}, \tag{4}$$

and the rotation matrix ${}^{i}\mathbf{R}_{i-1}$ which transfers a vector from ($\mathbf{x}_{i-1}$, $\mathbf{y}_{i-1}$, $\mathbf{z}_{i-1}$) coordinate system to ($\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$) coordinate system can be written as

$$^{i}\mathbf{R}_{i-1} = \begin{bmatrix} C\theta_i & S\theta_i & 0 \\ -C\alpha_{i,i-1}S\theta_i & C\alpha_{i,i-1}C\theta_i & S\alpha_{i,i-1} \\ S\alpha_{i,i-1}S\theta_i & -S\alpha_{i,i-1}C\theta_i & C\alpha_{i,i-1} \end{bmatrix}, \tag{5}$$

where $C\theta_i = \cos(\theta_i)$, $S\theta_i = \sin(\theta_i)$, $C\alpha_{i,i-1} = \cos(\alpha_{i,i-1})$ and $S\alpha_{i,i-1} = \sin(\alpha_{i,i-1})$.

From Eq. (5), it can be seen that the unit vector along the rotational axis of joint k represented in the ($\mathbf{x}_k$, $\mathbf{y}_k$, $\mathbf{z}_k$) coordinate system is a function of twist angle $\alpha_{k,k-1}$, i.e.,

$$^{k}\mathbf{z}_{k-1} = [0 \quad S\alpha_{k,k-1} \quad C\alpha_{k,k-1}]^{\mathrm{T}}. \tag{6}$$

From Eq. (1), the angular velocity of primary link i, referred to its own coordinate system, can be written as

$$^{i}\boldsymbol{\omega}_i = \sum_{k=1}^{i}({}^{i}\mathbf{z}_{k-1}\dot{\theta}_k), \tag{7}$$

where

$$^{i}\mathbf{z}_{k-1} = {}^{i}\mathbf{R}_{i-1}\cdots{}^{k+1}\mathbf{R}_k\,{}^{k}\mathbf{z}_{k-1}. \tag{8}$$

In Eq. (7), ${}^{i}\mathbf{z}_{k-1}\theta_k$ is called the partial angular velocity [8] of ${}^{i}\boldsymbol{\omega}_i$ associated with θ_k. From Eqs. (5), (6) and (8), it can be seen that the unit vector along the rotational axis of joint k represented in the ($\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$) coordinate system, ${}^{i}\mathbf{z}_{k-1}$, is a function of joint angles θ_p ($p = k+1,\ldots, i$). Let $\theta_1, \theta_2,\ldots, \theta_i$, the joint angles preceding θ_i, be the *inner joint angles* of

θ_i. From Eq. (7), it can be seen that the partial angular velocity of ${}^i\boldsymbol{\omega}_i$ associated with θ_k is a function of inner joint angles, θ_p ($p = k + 1, \ldots, i$) of θ_i.

4. Kinematics of Secondary Links

The fundamental circuit equations and coaxiality conditions [13] are used to derive angular displacements of secondary links. Let links j and k form a gear pair, and link i be the carrier. Then, links i, j and k form a fundamental circuit whose equation can be written as [6]

$$q_{j,i} = g_{k,j} q_{k,i} \tag{9}$$

where $g_{k,j} = \pm N_k/N_j$ is the gear ratio of the gear pair, positive or negative depending on a positive rotation of one gear producing a positive or negative rotation of the other about their pre-defined axes of rotation, and N_k is the teeth number of the gear attached to link k.

Let i, j and k be three coaxial links, the coaxial equation about relative motions among these three links can be written as

$$q_{i,k} = q_{i,j} + q_{j,k}. \tag{10}$$

From Eqs. (9) and (10), it can be shown that the relative angular velocity of a secondary link j with respect to its associated primary link i of an n-dof GRM can be written as

$$\dot{q}_{j,i} = \sum_{k=i+1}^{n} b_{j,k} \dot{\theta}_k \tag{11}$$

where $b_{j,k}$, coefficient of the relative angular velocity associated with θ_k, is a function of gear ratios.

Let joints $\theta_{i+1}, \theta_{i+2}, \ldots, \theta_n$, the joint angles succeeding θ_i, be the *outer joint angles* of θ_i and their time derivatives be the outer generalized speeds [8] of θ_i. From Eq. (11), it can be seen that the relative angular velocity of a secondary link j with respect to its associated primary link i is a function of outer generalized speeds of θ_i.

A collection of the time derivatives of $q_{j,i}$ for all input links forms the velocity vector of the actuator-space which can be written in terms of velocity vector of the joint space as

$$\dot{\boldsymbol{\Phi}} = \mathbf{A}^{\mathrm{T}} \dot{\boldsymbol{\Theta}} \tag{12}$$

where $\boldsymbol{\Phi}$ is the displacement vector associated with the actuator space, $\boldsymbol{\Theta} = [\theta_1\ \theta_2 \cdots \theta_n]^{\mathrm{T}}$ is the displacement vector associated with the joint space, $\mathbf{A}$ is the structure matrix [1], and $(\ )^{\mathrm{T}}$ denotes the transpose of ( ).

5. Generalized Inertia Force Contribution

The main idea for the dynamic modeling is based on the identification of primary and secondary links. All the secondary links contribute to the generalized inertia force in two parts. The first part is due to the motions of the primary links and the second part is due to their motions relative to the primary links which carry them. By the concept of virtual links, the first part of generalized inertia force can be computed incorporated with that of primary links. The second part of contribution can be derived by Lagrangian formulation separately. The generalized active force is then formulated and combined with the generalized inertia force to form the dynamic equations of motion.

5.1. Motions of primary links

Virtual link i^* can be obtained by treating all secondary links carried by primary link i as being rigidly attached to the primary link itself. A collection of virtual links forms a virtual EOLC. Hence, each virtual link i^* has the same motion with the primary link i and the same mass properties with primary link i and all secondary links carried by it. The angular velocities and mass center accelerations of virtual links can be calculated recursively in the same way as those for the open-loop chain by using Eqs. (1)–(3). Inertia properties of virtual links can be evaluated from those of associated secondary and primary links. The generalized inertia force contribution of the virtual EOLC, $\mathbf{G}_{\mathrm{v}}$, can be formulated by any of the methods used for open-loop manipulators [7, 10, 12, 14]:

$$\mathbf{G}_{\mathrm{v}} = \mathbf{M}_{\mathrm{v}}\ddot{\boldsymbol{\Theta}} + \mathbf{C}_{\mathrm{v}}(\boldsymbol{\Theta}, \dot{\boldsymbol{\Theta}}) \tag{13}$$

where $\mathbf{M}_{\mathrm{v}}$ is an $(n \times n)$ inertia coefficient matrix, and $\mathbf{C}_{\mathrm{v}}$, an $(n \times 1)$ vector, representing the Coriolis and centrifugal effects associated with $\mathbf{G}_{\mathrm{v}}$, respectively.

5.2. Relative motions of secondary links

Assuming axis–symmetric secondary links, the kinetic energy $\mathbf{K}_{j/i}$ due to the relative motion of secondary link j with respect to primary link i can be written as [4]

$$\mathbf{K}_{j/i} = \frac{1}{2} I_j \dot{q}_{j,i}^2 + I_j \dot{q}_{j,i}({}^i\mathbf{e}_j \cdot {}^i\boldsymbol{\omega}_i) \tag{14}$$

where I_j is the moment of inertia of secondary link j about ${}^i\mathbf{e}_j$, unit vector along its axis of rotation.

Substituting Eqs. (7) and (11) into Eq. (14), kinetic energy $\mathbf{K}_{j/i}$ of an n-dof GRM can be rewritten as

$$\mathbf{K}_{j/i} = \frac{1}{2} I_j \left[\sum_{k=i+1}^{n} (b_{j,k}\dot{\theta}_k) \right]^2 + I_j \sum_{k=i+1}^{n} (b_{j,k}\dot{\theta}_k) \left[\sum_{k=1}^{i} ({}^i\mathbf{e}_j \cdot {}^i\mathbf{z}_{k-1})\dot{\theta}_k \right]. \tag{15}$$

Applying a Lagrangian equation to Eq. (15), generalized inertia force contribution due to relative motion of secondary link j to primary link i associated with θ_{r}, $\gamma_{j/i,\mathrm{r}}$, can be derived as:

Where θ_r is an inner joint angle of θ_i:

$$\gamma_{j/i,r} = I_j({}^i\mathbf{e}_j \cdot {}^i\mathbf{z}_{r-1}) \sum_{k=i+1}^{n} (b_{j,k}\ddot{\theta}_k) - I_j \sum_{k=i+1}^{n} (b_{j,k}\dot{\theta}_k)\, {}^i\mathbf{e}_j \cdot \sum_{h=1}^{r} \frac{\partial {}^i\mathbf{z}_{h-1}}{\partial \theta_r}\dot{\theta}_h + I_j \sum_{k=i+1}^{n} b_{j,k}\dot{\theta}_k\, {}^i\mathbf{e}_j \cdot \sum_{h=r+1}^{i} \frac{\partial {}^i\mathbf{z}_{r-1}}{\partial \theta_h}\dot{\theta}_h, \tag{16a}$$

where θ_r is an outer joint angle of θ_i;

$$\gamma j/i, r = I_j b_{j,r}\left[\sum_{k=1}^{i}({}^i\mathbf{e}_j \cdot {}^i\mathbf{z}_{k-1})\ddot{\theta}_k + \sum_{k=i+1}^{n} b_{j,k}\ddot{\theta}_k\right] + I_j b_{j,r}\left[\sum_{k=1}^{i}\left({}^i\mathbf{e}_j \cdot \sum_{h=k}^{i} \frac{\partial {}^i\mathbf{z}_{k-1}}{\partial \theta_h}\dot{\theta}_h\right)\dot{\theta}_k\right]. \tag{16b}$$

The collection of $\gamma_{j/i,r}$ (r = 1 to n) forms the generalized inertia force $\boldsymbol{\gamma}_{j/i}$ as

$$\boldsymbol{\gamma}_{j/i} = \sum_{r=1}^{n}\sum_{k=1}^{n} \mathbf{M}_{j/i})_{r,k}\ddot{\theta}_k + \sum_{r=1}^{n}\left[\sum_{k=1}^{n}\sum_{h=1}^{n} \mathbf{C}_{j/i,r})_{h,k}\dot{\theta}_h\dot{\theta}_k\right] = \mathbf{M}_{j/i}\ddot{\boldsymbol{\Theta}} + \begin{bmatrix} \dot{\Theta}^{\mathrm{T}} C_{j/i,1}\dot{\Theta} \\ \vdots \\ \dot{\Theta}^{\mathrm{T}} C_{j/i,r}\dot{\Theta} \\ \vdots \\ \dot{\Theta}^{\mathrm{T}} C_{j/i,n}\dot{\Theta} \end{bmatrix} = \mathbf{M}_{j/i}\ddot{\boldsymbol{\Theta}} + \mathbf{C}_{j/i}(\boldsymbol{\Theta}, \dot{\boldsymbol{\Theta}}) \tag{17}$$

where $\boldsymbol{\gamma}_{j/i}$ is an $(n \times 1)$ vector, $=[\gamma_{j/i,1} \cdots \gamma_{j/i,r} \cdots \gamma_{j/i,n}]^{\mathrm{T}}$, $\mathbf{M}_{j/i}$ is an $(n \times n)$ symmetric inertia coefficient matrix and $\mathbf{C}_{j/i}$ is an $(n \times 1)$ vector specifying the Coriolis effect.

The $(r,\ k)$ element of $\mathbf{M}_{j/i}$, $\mathbf{M}_{j/i})_{r,k}$, the effective inertia associated with generalized coordinates, θ_r and θ_k, from Eq. (16)(a), (b), can be rearranged as follows:

where θ_r is an inner joint angle of θ_i:

$$\mathbf{M}_{j/i})_{r,k} = 0 \qquad \theta_k : \text{an inner joint angle of } \theta_i, \tag{18a}$$

$$= I_j b_{j,k}\, {}^i\mathbf{e}_j \cdot {}^i\mathbf{z}_{r-1} \qquad \theta_k : \text{an outer joint angle of } \theta_i, \tag{18b}$$

where θ_r is an outer joint angle of θ_i:

$$\mathbf{M}_{j/i})_{r,k} = I_j b_{jr}^{i}\mathbf{e}_j \cdot {}^i\mathbf{z}_{k-1} \qquad \theta_k : \text{an inner joint angle of } \theta_i, \tag{18c}$$

$$= I_j b_{jr} b_{j,k} \qquad \theta_k : \text{an outer joint angle of } \theta_i. \tag{18d}$$

Hence, $\mathbf{M}_{j/i})_{r,k}$ can be classified according to θ_r and θ_k being inner joints and/or outer joint angles of θ_i as:

(1). Where θ_r and θ_k are both inner joint angles of θ_i, $\mathbf{M}_{j/i})_{r,k}$ is equal to zero.

(2). Where θ_r is an inner joint angle of θ_i and θ_k is an outer joint angle of θ_i:

$\mathbf{M}_{j/i})_{r,k}$ is dependent on the projection of ${}^i\mathbf{e}_j$, onto ${}^i\mathbf{z}_{r-1}$, the unit vector along the rotational axis of inner joint angle θ_r. Note that, from Eq. (18)(b), (c), $\mathbf{M}_{j/i})_{r,k}$, where θ_r is an inner joint angle and θ_k an outer joint angle of θ_i, is the same as $\mathbf{M}_{j/i})_{r,k}$, where θ_r is an outer joint angle and θ_k an inner joint angle of θ_i.

(3). Where θ_r and θ_k are both outer joint angles of θ_i, $\mathbf{M}_{j/i})_{r,k}$ is independent of the joint angles.

The (h, k) element of $\mathbf{C}_{j/i,r}$, $\mathbf{C}_{j/i,r})_{h,k}$, the Coriolis effect associated with generalized coordinate θ_r and related to the time derivatives of θ_h and θ_k, from Eq. (16)(a), (b), can be rearranged as follows:

where θ_r is an inner joint angle of θ_i:

$$\mathbf{C}_{j/i,r})_{h,k} = -\frac{1}{2} I_j b_{j,k}\ {}^i\mathbf{e}_j \cdot \frac{\partial^i \mathbf{z}_{h-1}}{\partial\theta_r} \begin{cases} \theta_h : \text{a joint angle preceding } \theta_r \\ \theta_k : \text{an outer joint angle of } \theta_i \end{cases} \quad (19a)$$

$$= \frac{1}{2} I_j b_{j,k}\ {}^i\mathbf{e}_j \cdot \frac{\partial^i \mathbf{z}_{r-1}}{\partial\theta_h} \begin{cases} \theta_h : \text{an inner joint angle of } \theta_i \text{ and succeeding } \theta_r \\ \theta_k : \text{an outer joint angle of } \theta_i, \end{cases} \quad (19b)$$

$$= -\frac{1}{2} I_j b_{j,h}\ {}^i\mathbf{e}_j \cdot \frac{\partial^i \mathbf{z}_{k-1}}{\partial\theta_r} \begin{cases} \theta_h : \text{an outer joint angle of } \theta_i \\ \theta_k : \text{a joint angle preceding } \theta_r, \end{cases} \quad (19c)$$

$$= \frac{1}{2} I_j b_{j,h}\ {}^i\mathbf{e}_j \cdot \frac{\partial^i \mathbf{z}_{r-1}}{\partial\theta_k} \begin{cases} \theta_h : \text{an outer joint angle of } \theta_i \\ \theta_k : \text{an inner joint angle of } \theta_i \text{ and suceeding } \theta_r \end{cases} \quad (19d)$$

$$=0. \quad (19e)$$

Note that the partial derivative of ${}^i\mathbf{z}_{r-1}$ with respect to θ_h, $\partial^i\mathbf{z}_{r-1}/\partial\theta_h$, will be significant for θ_h is one among joint angles $(\theta_{r+1},\ldots,\theta_i)$, since ${}^i\mathbf{z}_{r-1}$ is a function of joint angles $\theta_{r+1},\ldots,\theta_i$. Hence, where θ_r is an inner joint angle of θ_i, $\mathbf{C}_{j/i,r})_{h,k}$ can be classified according to θ_h and θ_k being preceding or succeeding inner joint angles with an outer joint angle of θ_r:

(1). Where θ_h or θ_k precedes θ_r and the other is an outer joint angle of θ_i:

θ_h is a joint angle preceding θ_r, $\mathbf{C}_{j/i,r})_{h,k}$ is dependent of the projection of ${}^i\mathbf{e}_j$, onto $\partial\,{}^i\mathbf{z}_{h-1}/\partial\theta_r$, a partial derivative of the unit vector along the rotational axis of θ_h, ${}^i\mathbf{z}_{h-1}$, with respect to θ_r. Note that, from Eq. (19)(a), (c), $\mathbf{C}_{j/i,r})_{h,k}$, where θ_h is a joint angle preceding θ_r, and θ_k is an outer joint angle of θ_i, is the same as $\mathbf{C}_{j/i,r})_{h,k}$, where θ_k is a joint angle preceding θ_r, and θ_h an outer joint angle of θ_i.

(2). Where θ_h or θ_k succeeds θ_r and the other is an outer joint angle of θ_i:

θ_h is an inner joint of θ_i and also succeeds joint angle θ_r, $\mathbf{C}_{j/i,r})_{h,k}$ is dependent of the projection of ${}^i\mathbf{e}_j$, onto $\partial^i\,\mathbf{z}_{r-1}/\partial\theta_h$, the partial derivative of the unit vector along the rotational axis of θ_{r-1}, ${}^i\mathbf{z}_{r-1}$, with respect to θ_h. Note that, from Eq. (19)(b), (c), $\mathbf{C}_{j/i,r})_{h,k}$, where θ_h is an inner joint of θ_i and also succeeds joint angle θ_r, and θ_k is an outer joint angle of θ_i, is the same as $\mathbf{C}_{j/i,r})_{h,k}$, where θ_k is an inner joint of θ_i and also succeeds joint angle θ_r, and θ_h is an outer joint angle of θ_i.

(3). Where θ_h and θ_k are both inner joint angles or outer joint angles of θ_i, $\mathbf{C}_{j/i,r})_{h,k}$ is equal to zero.

Hence, where θ_r is an inner joint angle of θ_i, the elements of $\mathbf{C}_{j/i,r}$ are dependent on the partial derivatives of the unit vector along the rotational axis of the joint angles preceding θ_r.

Where θ_r is an outer joint angle of θ_i:

$$\mathbf{C}_{j/i,r})_{h,k} = \frac{1}{2} I_j b^i_{j,r} \mathbf{e}_j \cdot \frac{\partial^i \mathbf{z}_{h-1}}{\partial \theta_k} \begin{cases} \theta_h : \text{an inner joint angle of } \theta_i \text{ and preceding } \theta_k \\ \theta_k : \text{an inner joint angle of } \theta_i, \end{cases} \tag{20a}$$

$$= \frac{1}{2} I_j b^i_{j,r} \mathbf{e}_j \cdot \frac{\partial^i \mathbf{z}_{k-1}}{\partial \theta_h} \begin{cases} \theta_h : \text{an inner joint angle of } \theta_i \\ \theta_k : \text{an inner joint angle of } \theta_i \text{ an preceding } \theta_h \end{cases} \tag{20b}$$

$$=0. \tag{20c}$$

where θ_r is an outer joint angle of θ_i, $\mathbf{C}_{j/i,r})_{h,k}$ can also be classified according to θ_h and θ_k as preceding or succeeding inner joint angles and outer joint angles of θ_r:

(1) Where θ_h and θ_k are both inner joint angles of θ_i:

θ_h precedes θ_k, $\mathbf{C}_{j/i,r})_{h,k}$ is dependent on the projection of ${}^i\mathbf{e}_j$, onto $\partial^i\ \mathbf{z}\ _{h-1}/\partial\theta_k$, the partial derivative of the unit vector along the rotational axis of θ_h, ${}^i\ \mathbf{z}\ _{h-1}$, with respect to θ_k. Note that, from Eq. (20)(a), (c), $\mathbf{C}_{j/i,r})_{h,k}$, where θ_h is an inner joint angle of θ_i and also precedes joint angle θ_k, and θ_k is an inner joint angle of θ_i, is the same as $\mathbf{C}_{j/i,r})_{h,k}$, where θ_k is an inner joint of θ_i and also precedes joint angle θ_h, and θ_h is an outer joint angle of θ_i. For the diagonal elements of $\mathbf{C}_{j/i,r}$, $\partial^i\mathbf{z}\ _{h-1}/\partial\theta_k$ (for $h = k$) is equal to zero since the term ${}^i\mathbf{z}_{k-1}$ is a function of joint angles $(\theta_{k+1},\ldots, \theta_i)$.

(2) Where either θ_h or θ_k is an outer joint angle of θ_i, $\mathbf{C}_{j/i,r})_{h,k}$ is equal to zero.

From Eqs. (19) and (20), it can be seen that coupling matrices $\mathbf{C}_{j/i,r}$ (for $r = 1$ to n) are symmetric, and the elements are dependent on the partial derivatives of the unit vector along rotational axes of the inner joint angles of θ_i except ${}^i\mathbf{z}_{i-1}$ itself. For the generalized inertia force contribution $\boldsymbol{\gamma}_{j/i}$, the centrifugal effect does not exist since the diagonal elements of $\mathbf{C}_{j/i,r}$ are equal to zero. Hence, $\boldsymbol{\gamma}_{j/i}$ includeds only inertia and Coriolis effect.

Thus, the generalized inertia force contribution, due to relative motions of secondary links with respect to primary links which carry them, $\mathbf{G}_s$ can be obtained by collecting $\mathbf{M}_{j/i}$ and $\mathbf{C}_{j/i}$:

$$\mathbf{G}_s = \sum_{i=0}^{n}\sum_{j}\boldsymbol{\gamma}_{j/i} = \left[\sum_{i=0}^{n}\sum_{j}\mathbf{M}_{j/i}\right]\ddot{\boldsymbol{\Theta}} + \sum_{i=0}^{n}\sum_{j}\mathbf{C}_{j/i}(\boldsymbol{\Theta},\dot{\boldsymbol{\Theta}}) = \mathbf{M}_s\ddot{\boldsymbol{\Theta}} + \mathbf{C}_s(\boldsymbol{\Theta},\dot{\boldsymbol{\Theta}}) \tag{21}$$

where $\mathbf{M}_s$ is an $(n \times n)$ inertia coefficient matrix, and $\mathbf{C}_s$ is an $(n \times 1)$ vector specifying the Coriolis effects associated with $\mathbf{G}_s$.

6. Dynamic Equations of Motion

With the joints as the generalized coordinates, it has been shown that the generalized active forces, $\mathbf{G}$, are resultant joint torques and can be expressed as [2]

$$\mathbf{G} = \mathbf{A}\xi \tag{22}$$

where ξ is the input torque vector associated with the actuator space.

Equating the generalized inertia forces contribution $\mathbf{G}_v$ and $\mathbf{G}_s$ to the generalized active forces yields the equations of motion as

Table 1
D-H parameters

i	$a_{i,i-1}$	$\alpha_{i,i-1}$	$d_{i,i-1}$
1	0	$\pi/2$	d_1
2	a_2	0	0
3	0	$\pi/2$	0

$$(\mathbf{M}_v + \mathbf{M}_s)\ddot{\boldsymbol{\Theta}} + \mathbf{C}_v(\boldsymbol{\Theta}, \dot{\boldsymbol{\Theta}}) + \mathbf{C}_s(\boldsymbol{\Theta}, \dot{\boldsymbol{\Theta}}) = \mathbf{A}\xi. \tag{23}$$

7. A 3-dof GRM Example

Table 1 shows the D–H parameters of the example 3-dof GRM. From Eq. (4), we have ${}^1\mathbf{p}_{1,0} = [0\ d_1\ 0]^T$, ${}^2\mathbf{p}_{2,1} = [a_2\ 0\ 0]^T$ and ${}^3\mathbf{p}_{3,2} = [0\ 0\ 0]^T$. Let $\mathbf{r}_{1\ s} = [0\ -r_1\ 0]^T$, $\mathbf{r}_{2\ s} = [-r_2\ 0\ 0]^T$, $\mathbf{r}_{3\ s} = [0\ 0\ 0]^T$ and assume that ${}^0\boldsymbol{\omega}_0 = {}^0\mathbf{v}_0 = [0\ 0\ 0]^T$, and their time derivatives, = $[0\ 0\ 0]^T$. Using Eqs. (7) and (8), angular velocities of primary links 1, 2 and 3 can be computed as

$${}^1\boldsymbol{\omega}_1 = {}^1\mathbf{z}_0\dot{\theta}_1, \tag{24}$$

$${}^2\boldsymbol{\omega}_2 = {}^2\mathbf{z}_0\dot{\theta}_1 + {}^2\mathbf{z}_1\dot{\theta}_2, \tag{25}$$

$${}^3\boldsymbol{\omega}_3 = {}^3\mathbf{z}_0\dot{\theta}_1 + {}^3\mathbf{z}_1\dot{\theta}_2 + {}^3\mathbf{z}_2\dot{\theta}_3 \tag{26}$$

where the ${}^i\mathbf{z}_j$'s are shown in Table 2(a).

Table 2
(a) The orientation of rotation axes of joints

${}^1\mathbf{z}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	${}^2\mathbf{z}_0 = \begin{bmatrix} S\theta_2 \\ C\theta_2 \\ 0 \end{bmatrix}$	${}^3\mathbf{z} = \begin{bmatrix} S\theta_2 C\theta_3 + C\theta_2 S\theta_3 \\ 0 \\ S\theta_2 S\theta_3 - C\theta_2 C\theta_3 \end{bmatrix}$
${}^2\mathbf{z}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	${}^3\mathbf{z}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	${}^3\mathbf{z}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(b) Their partial derivatives

$\partial\, {}^2\mathbf{z}_0/\partial\theta_2 = \begin{bmatrix} C\theta_2 \\ -S\theta_2 \\ 0 \end{bmatrix}$	$\partial\, {}^3\mathbf{z}_0/\partial\theta_2 = \begin{bmatrix} C\theta_2 C\theta_3 - S\theta_2 S\theta_3 \\ 0 \\ C\theta_2 S\theta_3 + S\theta_2 C\theta_3 \end{bmatrix}$
$\partial\, {}^3\mathbf{z}_0/\partial\theta_3 = \begin{bmatrix} -S\theta_2 S\theta_3 + C\theta_2 C\theta_3 \\ 0 \\ S\theta_2 C\theta_3 + C\theta_2 S\theta_3 \end{bmatrix}$	$\partial\, {}^3\mathbf{z}_1/\partial\theta_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

From Fig. 2(b), it can be seen that virtual link 0* has the mass/inertia properties of primary link 0, virtual link 1* has the mass/inertia properties of primary link 1 and secondary link 6, virtual link 2* has the mass/inertia properties of primary link 2 and secondary link 7, and virtual link 3* has the mass/inertia properties as primary 3. With the mass properties of virtual links assumed as shown in Table 3, the generalized inertia force contribution due to the virtual EOLC, $\mathbf{G}_v$, is shown in the Appendix.

Using Eqs. (9) and (10), relative angular velocities of secondary links with respect to their associated primary links can be written as

$$\dot{q}_{4,0} = g_{1,4}\dot{\theta}_1, \tag{27}$$

$$\dot{q}_{5,0} = \dot{\theta}_1 + g_{2,5}\dot{\theta}_2, \tag{28}$$

$$\dot{q}_{6,1} = \dot{\theta}_2 + g_{7,6}g_{3,7}\dot{\theta}_3, \tag{29}$$

$$\dot{q}_{7,2} = g_{3,7}\dot{\theta}_3. \tag{30}$$

The generalized inertia force contribution due to relative motions of secondary links can be formulated according to the locations of their associated primary links as follows:

7.1. Associated with primary link 0

Where secondary link j is associated with primary link 0, there is no inner joint angle. Hence, elements of inertia matrix $\mathbf{M}_{j/0}$ can be derived from Eq. (18)d) while from Eq. (20)c), elements of matrices $\mathbf{C}_{j/0,r}$ (for r = 1 to 3) are equal to zero. For the example of a 3-dof GRM, secondary links 4 and 5 are carried by primary link 0. From Eqs. (27) and (28), we have $b_{4,2} = b_{4,3} = b_{5,3} = 0$, $b_{4,1} = g_{1,4}$, $b_{5,1} = 1$ and $b_{5,2} = g_{2,5}$. Thus, we have

Table 3
Inertia properties of virtual links

i^*	m_{i^*}	$\mathbf{r}_{i^*s}$	${}^{i^*}\mathbf{I}_{i^*}$
1*	m_{1*}	$\begin{bmatrix} 0 \\ r_{1*y} \\ r_{1*z} \end{bmatrix}$	$\begin{bmatrix} I_{1*,xx} & 0 & 0 \\ 0 & I_{1*,yy} & I_{1*,yz} \\ 0 & I_{1*yz} & I_{1*,zz} \end{bmatrix}$
2*	m_{2*}	$\begin{bmatrix} r_{2*x} \\ 0 \\ r_{2*z} \end{bmatrix}$	$\begin{bmatrix} I_{2*,xx} & 0 & I_{2*,xz} \\ 0 & I_{2*,yy} & 0 \\ I_{2*,xz} & 0 & I_{2*,zz} \end{bmatrix}$
3	m_3	$\begin{bmatrix} 0 \\ 0 \\ r_{3z} \end{bmatrix}$	$\begin{bmatrix} I_{3,xx} & 0 & 0 \\ 0 & I_{3,yy} & 0 \\ 0 & 0 & I_{3,zz} \end{bmatrix}$

$$\mathbf{M}_{4/0} = I_4 \begin{bmatrix} b_{4,1}^2 & b_{4,1}b_{4,2} & b_{4,1}b_{4,3} \\ b_{4,1}b_{4,2} & b_{4,2}^2 & b_{4,2}b_{4,3} \\ b_{4,1}b_{4,3} & b_{4,2}b_{4,3} & b_{4,3}^2 \end{bmatrix} = I_4 \begin{bmatrix} g_{1,4}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{31}$$

$$\mathbf{M}_{5/0} = I_5 \begin{bmatrix} b_{5,1}^2 & b_{5,1}b_{5,2} & b_{5,1}b_{5,3} \\ b_{5,1}b_{4,2} & b_{5,2}^2 & b_{5,2}b_{5,3} \\ b_{5,1}b_{5,3} & b_{5,2}b_{5,3} & b_{5,3}^2 \end{bmatrix} = I_5 \begin{bmatrix} 1 & g_{2,5} & 0 \\ g_{2,5} & g_{2,5}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{32}$$

and

$$\mathbf{C}_{4/0,1} = \mathbf{C}_{4/0,2} = \mathbf{C}_{4/0,3} = \mathbf{C}_{5/0,1} = \mathbf{C}_{5/0,2} = \mathbf{C}_{5/0,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{33}$$

7.2. Associated with primary link 1

Where secondary link j associated with primary link 1, θ_1 is an inner joint angle preceding θ_1 and θ_2 and θ_3 are outer joint angles of θ_1. Hence, for elements of $\mathbf{M}_{j/0}$, $\mathbf{M}_{j/1})_{1,1}$ can be determined from Eq. (18)a), $\mathbf{M}_{j/1})_{1,2}$ and $\mathbf{M}_{j/1})_{1,3}$ can be determined from Eq. (18)b), and $\mathbf{M}_{j/1})_{2,2}$, $\mathbf{M}_{j/1})_{2,3}$, and $\mathbf{M}_{j/1})_{3,3}$ can be determined from Eq. (18)d).

For elements of $\mathbf{C}_{j/1,1}$, from Eq. (19)e), $\mathbf{C}_{j/1,1})_{1,1}$, $\mathbf{C}_{j/1,1})_{2,2}$, $\mathbf{C}_{j/1,1})_{2,3}$, and $\mathbf{C}_{j/1,1})_{3,3}$ are equal to zero, and $\mathbf{C}_{j/1,1})_{1,2}$ and $\mathbf{C}_{j/1,1})_{1,3}$ can be determined from Eq. (19)a). Regarding elements of $\mathbf{C}_{j/1,2}$ and $\mathbf{C}_{j/1,3}$, from Eq. (20)a), $\mathbf{C}_{j/1,2})_{1,1}$ and $\mathbf{C}_{j/1,3})_{1,1}$ can be determined, as well as $\mathbf{C}_{j/1,2})_{1,2}$, while $\mathbf{C}_{j/1,2})_{2,2}$, $\mathbf{C}_{j/1,2})_{2,3}$, $\mathbf{C}_{j/1,1})_{3,3}$, $\mathbf{C}_{j/1,3})_{1,2}$, $\mathbf{C}_{j/1,3})_{2,2}$, $\mathbf{C}_{j/1,3})_{2,3}$ and $\mathbf{C}_{j/1,3})_{3,3}$ are equal to zero from Eq. (20)c).

For the example of a 3-dof GRM, secondary link 6 is associated with primary link 1. From Eq. (29), we have $b_{6,1}=0$, $b_{6,2}=1$ and $b_{6,3}=g_{7,6}g_{3,7}$. From Figs. 1 and 3, ${}^1\mathbf{e}_6=[0\ 0\ 1]^{\mathrm{T}}$ and from Table 2, we have

$${}^1\mathbf{e}_6 \cdot {}^1\mathbf{z}_0 = {}^1 bfe_6 \cdot \frac{\partial {}^1\mathbf{z}_0}{\partial \theta_1} = 0. \tag{34}$$

Hence, we have

$$\mathbf{M}_{6/1} = I_6 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & g_{7,6}g_{3,7} \\ 0 & g_{7,6}g_{3,7} & g_{7,6}^2 g_{3,7}^2 \end{bmatrix}, \tag{35}$$

$$\mathbf{C}_{6/1,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{36}$$

and

$$\mathbf{C}_{6/1,2} = \mathbf{C}_{6/1,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{37}$$

7.3. Associated with primary link 2

Where secondary link j is associated with primary link 2, θ_1 and θ_2 are inner joint angles and θ_3 is an outer joint angle of θ_2. Note that θ_1 is an inner joint angle preceding θ_1 and θ_2, and θ_2 is an inner joint angle succeeding θ_1. Hence, $\mathbf{M}_{j/2})_{1,1}$, $\mathbf{M}_{j/2})_{1,2}$ and $\mathbf{M}_{j/2})_{2,2}$ can be determined from Eq. (18)a), $\mathbf{M}_{j/2})_{1,3}$ and $\mathbf{M}_{j/2})_{2,3}$ can be determined from Eq. (18)b), while $\mathbf{M}_{j/2})_{3,3}$ can be determined from Eq. (18)d). For elements of $\mathbf{C}_{j/2,1}$, from euation (19e), $\mathbf{C}_{j/2,1})_{1,1}$, $\mathbf{C}_{j/2,1})_{1,2}$, $\mathbf{C}_{j/2,1})_{2,2}$ and $\mathbf{C}_{j/2,1})_{3,3}$ are equal to zero, and $\mathbf{C}_{j/2,1})_{1,3}$ can be determined from Eq. (19)a) for θ_1 precedes θ_1, and $\mathbf{C}_{j/2,1})_{2,3}$ can be determined from Eq. (19)b) since θ_2 succeeds θ_1.

Where elements of $\mathbf{C}_{j/2,2}$, $\mathbf{C}_{j/2,2})_{1,1}$, $\mathbf{C}_{j/2,2})_{1,2}$, $\mathbf{C}_{j/2,2})_{2,2}$ and $\mathbf{C}_{j/2,2})_{3,3}$ are equal to zero from Eq. (19)e), $\mathbf{C}_{j/2,2})_{1,3}$ and $\mathbf{C}_{j/2,2})_{2,3}$ can be determined from Eq. (19)a) since θ_1 and θ_2 precede θ_2. Where elements of $\mathbf{C}_{j/2,3}$, $\mathbf{C}_{j/2,3})_{1,1}$, $\mathbf{C}_{j/2,3})_{1,2}$ and $\mathbf{C}_{j/2,3})_{2,2}$ can be determined from Eq. (20)a), $\mathbf{C}_{j/2,3})_{1,3}$, $\mathbf{C}_{j/2,3})_{2,3}$ and $\mathbf{C}_{j/2,3})_{3,3}$ are equal to zero from Eq. (20)c).

For the example, in 3-dof GRM, the secondary link 7 is carried by primary link 2. From Eq. (30), we have $b_{7,1}=b_{7,2}=0$ and $b_{7,3}=g_{3,7}$. From Figs. 1 and 3, ${}^2\mathbf{e}_7=[1\ 0\ 0]^{\mathrm{T}}$ and from Table 2, we have

$${}^2\mathbf{e}_7 \cdot {}^2\mathbf{z}_0 = S\theta_2, \tag{38a}$$

$${}^2\mathbf{e}_7 \cdot \frac{\partial^2 \mathbf{z}_0}{\partial \theta_2} = \mathbf{C}\theta_2 \tag{38b}$$

and

$${}^2\mathbf{e}_7 \cdot {}^2\mathbf{z}_1 = {}^2\mathbf{e}_7 \cdot \frac{\partial^2 \mathbf{z}_1}{\partial \theta_2} = 0. \tag{38c}$$

Thus, we have

$$\mathbf{M}_{7/2} = I_7 \begin{bmatrix} 0 & 0 & g_{3,7}S\theta_2 \\ 0 & 0 & 0 \\ g_{3,7}S\theta_2 & 0 & g_{3,7}^2 \end{bmatrix}, \tag{39}$$

$$\mathbf{C}_{7/2,1} = \frac{1}{2} I_7 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & g_{3,7}C\theta_2 \\ 0 & g_{3,7}C\theta_2 & 0 \end{bmatrix}, \tag{40}$$

$$\mathbf{C}_{7/2,2} = -\frac{1}{2} I_7 \begin{bmatrix} 0 & 0 & g_{3,7}C\theta_2 \\ 0 & 0 & 0 \\ g_{3,7}C\theta_2 & 0 & 0 \end{bmatrix} \tag{41}$$

and

$$\mathbf{C}_{7/2,3} = \frac{1}{2} I_7 \begin{bmatrix} 0 & g_{3,7}C\theta_2 & 0 \\ g_{3,7}C\theta_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{42}$$

Where in Eqs. (1) and (40)–(42), $\mathbf{C}_{7/2}$ can be written as

$$\mathbf{C}_{7/2} \begin{bmatrix} \dot{\boldsymbol{\Theta}}^{\mathrm{T}} \mathbf{C}_{7/2,1} \dot{\boldsymbol{\Theta}} \\ \dot{\boldsymbol{\Theta}}^{\mathrm{T}} \mathbf{C}_{7/2,2} \dot{\boldsymbol{\Theta}} \\ \dot{\boldsymbol{\Theta}}^{T} \mathbf{C}_{7/2,3} \dot{\boldsymbol{\Theta}} \end{bmatrix} = \begin{bmatrix} I_7 g_{3,7} C\theta_2 \dot{\theta}_2 \dot{\theta}_3 \\ -I_7 g_{3,7} C\theta_2 \dot{\theta}_1 \dot{\theta}_3 \\ I_7 g_{3,7} C\theta_2 \dot{\theta}_1 \dot{\theta}_2 \end{bmatrix}. \tag{43}$$

Thus, by collecting $\mathbf{M}_{j/i}$'s and $\mathbf{C}_{j/i}$'s, generalized inertia force contribution $\mathbf{G}_{\mathrm{s}}$ can be written as

$$\mathbf{G}_{\mathrm{s}} = \begin{bmatrix} I_4 g_{1,4}^2 + I_5 & I_5 g_{2,5} & I_7 g_{3,7} S\theta_2 \\ I_5 g_{2,5} & I_5 g_{2,5}^2 + I_6 & I_6 g_{7,6} g_{3,7} \\ I_7 g_{3,7} S\theta_2 & I_6 g_{7,6} g_{3,7} & I_6 g_{7,6}^2 g_{3,7}^2 + I_7 g_{3,7}^2 \end{bmatrix} \ddot{\boldsymbol{\Theta}} + \begin{bmatrix} I_7 g_{3,7} C\theta_2 \dot{\theta}_2 \dot{\theta}_3 \\ -I_7 g_{3,7} C\theta_2 \dot{\theta}_1 \dot{\theta}_3 \\ I_7 g_{3,7} C\theta_2 \dot{\theta}_1 \dot{\theta}_2 \end{bmatrix}. \tag{44}$$

From Eqs. (12) and (27)–(29), structure matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = \begin{bmatrix} b_{4,1} & b_{5,1} & b_{6,1} \\ b_{4,2} & b_{5,2} & b_{6,2} \\ b_{4,3} & b_{5,3} & b_{6,3} \end{bmatrix} = \begin{bmatrix} g_{1,4} & 1 & 0 \\ 0 & g_{2,5} & 1 \\ 0 & 0 & g_{7,6} g_{3,7} \end{bmatrix}. \tag{45}$$

Hence, by equating the generalized inertia forces to the generalized active force, the equation of motion can be written as follows from Eq. (23):

$$\mathbf{G} = \mathbf{G}_{\mathrm{v}} + \mathbf{G}_{\mathrm{s}} = \mathbf{A}\boldsymbol{\xi} \tag{46}$$

where ξ_i is the input torque associated with input link i and $\boldsymbol{\xi} = [\xi_4 \ \xi_5 \ \xi_6]^{\mathrm{T}}$.

8. Summary

A methodology for the formulation of dynamic equations of geared robotic mechanisms has been developed in a systematic and concise manner. It is shown that the generalized inertia force contribution of the manipulator can be formulated by a two-step approach. With the concept of virtual links and virtual EOLC, contribution due to the motions of primary links is formulated consistent with that for open-loop manipulators. Contribution due to relative motions of secondary links are then formulated individually according to the locations of the primary links sequentially from base link to the end–effector. The generalized active forces are then formulated and combined with the generalized inertia forces to form the equations of motion. Thus, the coupling effect of the drive trains to the motion equation of the system can be identified. This approach can be implemented to an automated analysis algorithm and lead to a rational methodology for the driving train design of a general class of geared robotic mechanisms.

Appendix A

For a 3-link virtual EOLC with the mass properties as shown in Table 3, the generalized inertia force contribution can be shown as

$$\mathbf{G}_v = \begin{bmatrix} \phi_1 & \phi_2 & 0 \\ \phi_2 & \phi_3 & \phi_4 \\ 0 & \phi_4 & \phi_5 \end{bmatrix} \ddot{\boldsymbol{\Theta}} + \begin{bmatrix} \phi_6\dot{\theta}_2^2 + \phi_7\dot{\theta}_1\dot{\theta}_2 + \phi_8\dot{\theta}_1\dot{\theta}_3 \\ \phi_9\dot{\theta}_1^2 + \phi_{10}\dot{\theta}_3^2 + 2\phi_{10}\dot{\theta}_2\dot{\theta}_3 \\ \phi_{11}\dot{\theta}_1^2 + \phi_{10}\dot{\theta}_2^2/2 \end{bmatrix} \tag{A1}$$

where

$$\phi_1 = \phi_{12} + m_{2*}[\phi_{13}^2(C\theta_2)^2 + r_{2*z}^2] + I_{2*,xx} + [m_3\phi_{14}^2 + I_{3,xx}\phi_{17}^2 + I_{3,zz}\phi_{18}^2] \tag{A2a}$$

$$\phi_2 = (I_{2*,xx} - m_{2*}\phi_{13}r_{2*z})S\theta_2, \tag{A2b}$$

$$\phi_3 = [m_{2*}\phi_{13}^2 + I_{2*,zz}] + [m_3\phi_{14} + I_{3,yy}], \tag{A2c}$$

$$\phi_4 = m_3r_{3z}(a_2S\theta_3 + r_{3z}), \tag{A2d}$$

$$\phi_5 = m_3r_{3z} + I_{3,yy}, \tag{A2e}$$

$$\phi_6 = (-m_{2*}\phi_{13}r_{2*z} + I_{2*,xz}S\theta_2)C\theta_2, \tag{A2f}$$

$$\phi_7 = 2(-m_{2*}\phi_2 + \phi_{16})S\theta_2C\theta_2 - 2m_3\phi_{14}(a_2S\theta_2 + r_{3z}\phi_{18}) - 2\phi_{14}\phi_{15}\phi_{17}, \tag{A2g}$$

$$\phi_8 = -2m_3\phi_{18}r_{3z} + 2\phi_{14}\phi_{15}\phi_{17}, \tag{A2h}$$

$$\phi_9 = (m_{2*}\phi_{13}^2 - \phi_{16})S\theta_2C\theta_2 + m_3\phi_{14}(a_2S\theta_2 - r_{3z}\phi_{17}) + \phi_{14}\phi_{15}\phi_{17}, \tag{A2i}$$

$$\phi_{10} = m_3r_{3z}a_2C\theta_3, \tag{A2j}$$

$$\phi_{11} = m_3r_{3z}\phi_{17}\phi_{18} + \phi_{14}\phi_{15}\phi_{17} \tag{A2k}$$

and where

$$\phi_{12} = m_{1*} r_{1*z}^2 + I_{1*,yy}, \quad \text{(A3a)}$$

$$\phi_{13} = a_2 + r_{2*x}, \quad \text{(A3b)}$$

$$\phi_{14} = a_2 C\theta_2 + r_{3z}\phi_{17}, \quad \text{(A3c)}$$

$$\phi_{15} = I_{3,xx} - I_{3,yy}, \quad \text{(A3d)}$$

$$\phi_{16} = I_{2*,xx} - I_{2*,yy}, \quad \text{(A3e)}$$

$$\phi_{17} = S\theta_2 C\theta_3 + C\theta_2 S\theta_3, \quad \text{(A3f)}$$

$$\phi_{18} = S\theta_2 S\theta_3 - C\theta_2 C\theta_3. \quad \text{(A3g)}$$

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