A Decomposition Approach for the Kinematic Synthesis of Tendon-Driven Manipulators

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This article describes a decomposition methodology for the kinematic synthesis of tendon-driven manipulators (TDMs). Based on the QR factorization, the complex transformation between vectors in the tendon-space and the joint-space of an n-DOF TDM with $n+1$ tendons is decomposed as a two-step transformation. An orientation matrix is used to characterize the vector transformation between the $(n+1)$-dimensional tendon-space and the n-dimensional intermediate equivalent tendon-space. An equivalent structure matrix is also introduced for the vector transformation between the n-dimensional equivalent tendon-space and n-dimensional joint-space. Design equations for synthesizing a TDM to possess kinematic isotropic transmission characteristics with proper tendon routing and pulley sizes are derived.

INTRODUCTION

Tendon has been widely used for the power transmission system in robot manipulators. Since it allows the actuators to be installed as close to the base link as possible, the size and inertia of the manipulating system can be largely reduced. Tendon-driven manipulators (TDMs), such as the UTAH/MIT hand[1,2] and the Stanford/JPL hand,[3] have the advantage of remote control and light weight. A special characteristic associated with a TDM is its capability of supporting tension but not compression. Due to this distinct characteristic, a minimum of $n+1$ tendons is necessary for controlling an n degree of freedom (DOF) TDM.[3,4]

Lee and Tsai[5] developed a methodology for the enumeration of admissible pseudotriangular structure matrices for the tendon routing of TDMs by assuming the pulleys mounted on each joint axis are with the same size. Many researchers defined

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the condition number as the ratio of the maximal singular value to the minimal singular value of the Jacobian matrix to evaluate and/or design a manipulator.[3,6–9] Lee and Tsai[10] defined the condition number of a TDM as the ratio of the maximal singular value to the minimal singular value of the structure matrix. Ou and Tsai[11] extended the concept of the overall transformation matrix defined by Chen and Tsai[12] for articulated gear mechanisms to TDMs. Condition for TDMs with structure matrices of general form to possess isotropic transmission characteristics are derived based on the result of structure matrices with pseudotriangular form. Ou and Tsai[13] also showed that kinematic isotropic transmission characteristics can be achieved only if the n-DOF manipulator is driven by either $n+1$ or $2n$ tendons.

In this article, the vector transformation between $(n+1)$-dimensional tendon-space and n-dimensional joint space is treated as a two-step transformation by using the concept of QR factorization.[14] It will be shown that an equivalent structure matrix and an orientation matrix can be established to simplify the complex vector transformation between tendon-space and joint-space. Characteristics of these two matrices are derived and a systematic methodology to enumerate admissible orientation matrices is established. Necessary condition to determine the orientation and equivalent structure matrices are derived. Thus, appropriate tendon routing and pulley sizes for an n-DOF TDM with $n+1$ tendons with kinematic isotropic transmission characteristics can be systematically determined at a specified posture. A 2-DOF TDM is used as an illustrative example.

STRUCTURAL CHARACTERISTICS

Figure 1 shows the schematic of a general n-DOF TDM with $n+1$ actuating tendons. The joints are numbered sequentially from the base link to the distal end of the manipulator. The vector relations between the end-effector-space and the joint-space can be written as

$$\dot{\mathbf{X}} = \mathbf{J}\dot{\Theta} \tag{1}$$

and

$$\tau = \mathbf{J}^T \mathbf{F} \tag{2}$$

where $\mathbf{X}$ and Θ are $n \times 1$ displacement vectors at the end-effector-space and at the joint-space, respectively, $\mathbf{J}$ is an $n \times n$ Jacobian matrix, τ is an $n \times 1$ joint torque vector, and $\mathbf{F}$ is an $n \times 1$ general external force vector.

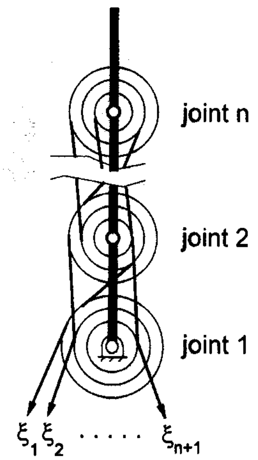

Figure 1. A general n-DOF TDM with $n+1$ tendons.

The vector relation between the tendon-space and joint-space can be expressed as[5]

$$\mathbf{S} = \mathbf{A}^T \Theta \tag{3}$$

and

$$\tau = \mathbf{A}\xi \tag{4}$$

where $\mathbf{S}$ is an $(n+1) \times 1$ tendon displacement vector, ξ is an $(n+1) \times 1$ tendon force vector, $\mathbf{A}$ is an $n \times (n+1)$ structure matrix, and $(\ )^T$ denotes the transpose of ( ).

Since the structure matrix $\mathbf{A}$ is not a square matrix, it can be seen that, given a set of join torques, tendon forces cannot be uniquely determined. There are infinite sets of solutions for the tendon forces associated with a set of desired joint torques. Thus, the inverse transformation of Eq. (4) can be written as[14]

$$\xi = \mathbf{A}^+ \tau + \lambda \mathbf{H}_A \tag{5}$$

where $\mathbf{A}^+$ is the pseudo-inverse of $\mathbf{A}$, $\mathbf{H}_A$ is the $(n+1) \times 1$ null space vector of $\mathbf{A}$, which denotes the pretension of tendons[11] and λ is an arbitrary nonzero constant.

Table I shows admissible 2- and 3-DOF structure matrices of TDMs. In Table I, nonzero elements are denoted by "#" and matrices are arranged according to the distribution of the tendon actuators. The letters "g", "s", and "e" denote that the loca-

Table I. Admissible structure matrices of TDMs.

(a) 2-DOF TDMs

$\begin{bmatrix} \# & \# & \# \\ \# & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# \\ \# & \# & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 \\ \# & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 \\ \# & 0 & \# \end{bmatrix}$
g^3-1	g^3-2	g^2s-1	g^2s-2

(b) 3-DOF TDMs

$\begin{bmatrix} \# & \# & \# & \# \\ \# & \# & \# & \# \\ \# & \# & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & \# \\ \# & \# & \# & \# \\ \# & \# & \# & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & \# \\ \# & \# & \# & 0 \\ \# & \# & \# & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & \# \\ \# & \# & \# & \# \\ \# & \# & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & \# \\ \# & \# & \# & 0 \\ \# & \# & 0 & 0 \end{bmatrix}$
g^4-1	g^4-2	g^4-3	g^4-4	g^4-5
$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & \# & \# \\ \# & \# & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & \# & \# \\ \# & \# & \# & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & \# & \# \\ \# & \# & 0 & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & \# & \# \\ \# & \# & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & \# & \# \\ \# & 0 & 0 & \# \end{bmatrix}$
g^3s-1	g^3s-2	g^3s-3	g^3s-4	g^3s-5
	$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & 0 & \# \\ \# & \# & 0 & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & 0 & \# \\ \# & \# & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & 0 & \# \\ \# & 0 & 0 & \# \end{bmatrix}$	
	g^3s-6	g^3s-7	g^3s-8	
$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & \# & 0 \\ \# & \# & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & \# & 0 \\ \# & \# & 0 & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & 0 & 0 \\ \# & \# & 0 & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & \# & 0 \\ \# & 0 & 0 & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & 0 & 0 \\ \# & 0 & 0 & \# \end{bmatrix}$
g^3e-1	g^3e-2	g^3e-3	g^3e-4	g^3e-5
$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & \# & \# & \# \\ \# & \# & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & \# & \# & \# \\ \# & \# & \# & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & \# & \# & \# \\ \# & 0 & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & \# & \# & \# \\ \# & 0 & \# & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & \# & \# & \# \\ 0 & 0 & \# & \# \end{bmatrix}$
g^2s^2-1	g^2s^2-2	g^2s^2-3	g^2s^2-4	g^2s^2-5
	$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & 0 & \# & \# \\ \# & 0 & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & 0 & \# & \# \\ \# & 0 & \# & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & 0 & \# & \# \\ 0 & 0 & \# & \# \end{bmatrix}$	
	g^2s^2-6	g^2s^2-7	g^2s^2-8	
$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & \# & \# & 0 \\ \# & \# & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & \# & \# & 0 \\ \# & \# & 0 & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & \# & \# & 0 \\ \# & 0 & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & 0 & \# & 0 \\ \# & 0 & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & \# & \# & 0 \\ \# & 0 & 0 & \# \end{bmatrix}$
g^2se-1	g^2se-2	g^2se-3	g^2se-4	g^2se-5
	$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & \# & \# & 0 \\ 0 & 0 & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & 0 & \# & 0 \\ 0 & 0 & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & 0 & \# & 0 \\ \# & 0 & 0 & \# \end{bmatrix}$	
	g^2se-6	g^2se-7	g^2se-8	

tion of the actuators are located on the ground, shoulder, and elbow axis, respectively, and the power stands for the number of tendon actuators installed in the joint axis. The characteristics of the structure matrices can be summarized as follows[11]:

C1. The rank of **A** is n.

C2. Elements in the null vector of **A** are of the same sign and not equal to zero.

C3. Nonzero elements in each column of **A** are consecutive, since tendons are routed from joint to joint in a continuous manner.

C4. The absolute value of the (i, j) element of **A** is equal to the radius of the pulley mounted on the ith joint and routed by the jth tendon.

Ou and Tsai[11] and Tsai[15] showed that a TDM can be designed to possess isotropic transmission characteristics at a given posture if the structure and Jacobian matrices satisfy the following two equations,

$$\mathbf{A}\mathbf{A}^T = \frac{1}{\mu^2}\mathbf{J}^T\mathbf{J} \tag{6}$$

and

$$\mathbf{A}\mathbf{H}_m = 0 \tag{7}$$

where $\mathbf{H}_m = [1, 1, \ldots, 1]^T$ is an $(n+1) \times 1$ column vector, and μ is a positive constant.

Note that for an n-DOF TDM with $n+1$ tendons, the number of equation available in Eq. (6) is $n(n+1)/2$ while that of Eq. (7) is n. Thus, the total number of equations available in determining the tendon routing and pulley sizes is $n(n+3)/2$. Hence, in general, the structure matrices shown in Table I with less than five unknowns for 2-DOF TDMs and with less than nine unknowns for 3-DOF TDMs are not possible to possess isotropic transmission characteristics at a given posture. On the other hand, the maximum number of unknowns in the structure matrix **A** is equal to $n(n+1)$. Hence, the elements of the structure matrix **A** can be solved with maximum $n(n-1)/2$ free choices. For a TDM having a pseudotriangular structure matrix, the number of unknowns in the structure matrix **A** is equal to $n(n+3)/2$, i.e., the number of unknowns is equal to that of equations. Hence, Ou and Tsai[11] used Eqs. (6) and (7) as the design equations to determine the tendon routing and pulley sizes for TDMs having a pseudotriangular structure matrix. They also extended the above equations to TDMs with general structure matrices. However, once the elements of the structure matrix are solved, the structure matrix may not necessarily satisfy condition C3. Also note that Eqs. (6) and (7) do not guarantee a valid solution for a desired structure matrix at a specified posture. The compatibility problem between the structure and Jacobian matrices must be addressed.[16] Table II shows the structure matrices with which kinematic isotropic condition can be obtained if none of the elements in the product of Jacobian matrix $\mathbf{J}^T\mathbf{J}$ is equal to zero at given posture.

Table II. Admissible structure matrices with which kinematic isotropic transmission characteristics can be obtained.

Cases	Nonzero $\mathbf{J}^T\mathbf{J}(i, j)$
(a) 2-DOF structure matrices	
g^3	g^3-1 g^3-2
g^2s	g^2s-1
(b) 3-DOF structure matrices	
g^4	g^4-1 g^4-2 g^4-3 g^4-4 g^4-5
g^3s	g^3s-1 g^3s-2 g^3s-3 g^3s-4 g^3s-5 g^3s-6
g^3e	g^3e-1 g^3e-2
g^2s^2	g^2s^2-1 g^2s^2-2 g^2s^2-3
g^2se	g^2se-1

EQUIVALENT TENDON SPACE

From C1 to C3, it can be seen that matrix **A** has independent columns. Using the QR factorization,[14] **A** can be expressed as

$$\mathbf{A} = (A^T)^T = (\tilde{\mathbf{Q}}\tilde{\mathbf{R}})^T = \mathbf{B}\mathbf{P} \tag{8}$$

where **B** is an $n \times n$ lower triangular matrix with positive diagonal elements and **P** is an $n \times (n+1)$ unitary matrix.

Substituting Eq. (8) into Eq. (4), yields

$$\tau = \mathbf{B}\eta \tag{9}$$

where

$$\eta = \mathbf{P}\xi \tag{10}$$

and where η is an $n \times 1$ column vector.

Let η be the equivalent tendon forces in the equivalent tendon-space. Equation (9) defines the transformation between the joint torques and equivalent tendon forces while Eq. (10) defines that between the equivalent tendon forces and tendon forces. Figure 2 shows the conceptual block diagram of a TDM.

Equivalent Structure Matrix

Equation (9) transforms the equivalent tendon forces into joint torques. Since matrix **B** is a square matrix,

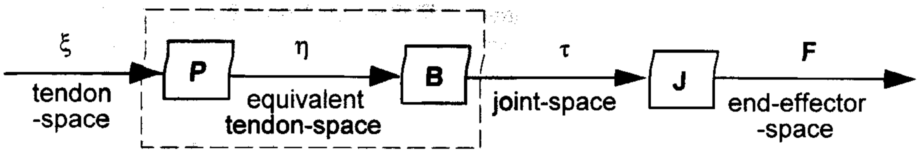

Figure 2. Conceptual diagram of a TDM.

given a set of desired joint torques, the equivalent tendon forces can be uniquely determined. Hence, the characteristics of matrix **B** is similar to the structure matrix defined by Chang and Tsai[17] for n-DOF articulated gear mechanisms with n input actuators and is called the equivalent structure matrix. The characteristics of equivalent structure matrix **B** of an n-DOF TDM can be summarized as follows.

D1. The matrix **B** is an $n \times n$ lower triangular matrix with nonzero determinant.

D2. The ith row of **B** describes how the resultant joint torques at joint i is affected by the equivalent tendon forces while the jth column describes how the jth equivalent tendon force is transmitted to various joints of the TDM. Hence, the columns of **B** represent the mechanical transmission lines[17] of the equivalent tendon system.

Determination of Equivalent Structure Matrix

Since matrix **P** is a unitary matrix, from Eq. (8), we have

$$\mathbf{A}\mathbf{A}^T = \mathbf{B}\mathbf{P}\mathbf{P}^T\mathbf{B}^T = \mathbf{B}\mathbf{B}^T \tag{11}$$

Substituting Eq. (11) into Eq. (6) yields

$$\mathbf{B}\mathbf{B}^T = \frac{1}{\mu^2}\mathbf{J}^T\mathbf{J} \tag{12}$$

Equation (12) is a sufficient condition for a TDM to possess kinematic isotropic transmission characteristics between the joint-space and equivalent tendon-space. The number of equations available in Eq. (12) is $n(n+1)/2$ for an n-DOF TDM, while the number of unknowns in equivalent structure matrix **B** is $n(n+1)/2$. Hence, the elements in the equivalent structure matrix **B** can be uniquely determined once the posture of the TDM is specified, i.e., the Jacobian matrix is given.

Orientation Matrix

Since matrix **P** is not a square matrix, the tendon forces cannot be uniquely determined by a given set of equivalent tendon forces. Note that row vectors of **P** are orthonormal and form a linearly independent base. Matrix **P** transfers the $(n+1)$-dimensional tendon forces into the n-dimensional equivalent tendon forces and is called the orientation matrix. The characteristics of orientation matrix **P** can be summarized as:

E1. The matrix **P** is an $n \times (n+1)$ unitary matrix with rank equal to n.

E2. The first nonzero element of each column in matrix **P** denotes the location of equivalent tendon actuator.

E3. Elements in the null space of **P** are of the same sign and not equal to zero (proved in the Appendix).

E4. The (i, j) element of **P**, p_{ij}, represents the direction cosine between the ith equivalent tendon force and the jth tendon force. $p_{ij} = 0$ implies that the jth tendon force is perpendicular to the ith equivalent actuator force. $p_{ij} \neq 0$ shows that the jth equivalent tendon force is affected by the ith tendon force with p_{ij} as a proportional constant.

Enumeration of Orientation Matrix

A feasible $n \times (n+1)$ orientation matrix has to satisfy the following conditions:

R1. There is a minimum of two nonzero elements in each row of **P**. This guarantees that every joint can be manipulated on both directions.

R2. The submatrix obtained by removing any column from **P** is nonsingular.

R3. The locations of zero elements in any two rows of matrix **P** are constrained such that it is possible to have the dot operation between these two row vectors equal to zero.

R4. Switching any two columns of an orientation matrix does not change the function and structure of the system.

Applying the above rules, admissible orientation matrices can be enumerated systematically. Admissible orientation matrices for 2-DOF TDMs are shown in Table IIIa while those for 3-DOF TDM are shown in Table IIIb. In Table III, nonzero elements are denoted by # and matrices are arranged according to the distribution of the equivalent tendon actuators. The letters "1", "2", and "3" denote that the locations of the actuators are located on the first, second, and third joint axes, respectively, and the power stands for the number of equivalent tendon actuators installed in the joint axis.

Determination of μ

From Eq. (4), the norm of joint torque vector τ can be written as

$$|\tau|^2 \equiv \tau^T\tau = \xi^T\mathbf{A}^T\mathbf{A}\,\xi \tag{13}$$

Hence, at a given posture, $|\tau|^2 = 1$ yields a force ellipsoid in the actuator-space. The force capacity, FC, is defined to be proportional to the volume of the ellipsoid, i.e.,

$$\mathrm{FC} \equiv \frac{1}{\prod_{i=1}^{n}\sqrt{v_i}} \tag{14}$$

Table III. Admissible orientation matrices of TDMs.

(a) 2-DOF TDMs

$\begin{bmatrix} \# & \# & \# \\ \# & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# \\ \# & \# & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 \\ \# & \# & \# \end{bmatrix}$
1^3-1	1^3-2	$1^2 2$-1

(b) 3-DOF TDMs

$\begin{bmatrix} \# & \# & \# & \# \\ \# & \# & \# & \# \\ \# & \# & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & \# \\ \# & \# & \# & \# \\ \# & \# & \# & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & \# \\ \# & \# & \# & 0 \\ \# & \# & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & \# \\ \# & \# & \# & 0 \\ \# & \# & \# & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & \# \\ \# & \# & \# & \# \\ \# & \# & 0 & 0 \end{bmatrix}$
1^4-1	1^4-2	1^4-3	1^4-4	1^4-5
$\begin{bmatrix} \# & \# & \# & \# \\ \# & \# & 0 & 0 \\ \# & \# & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & \# \\ \# & \# & 0 & \# \\ \# & \# & \# & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & \# \\ \# & \# & \# & 0 \\ \# & \# & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & \# \\ \# & \# & 0 & 0 \\ \# & \# & \# & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & \# \\ 0 & 0 & \# & \# \\ \# & \# & 0 & 0 \end{bmatrix}$
1^4-6	1^4-7	1^4-8	1^4-9	1^4-10
$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & \# & \# \\ \# & \# & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & \# & \# \\ \# & \# & \# & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & 0 & \# \\ \# & \# & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & \# & \# \\ \# & \# & 0 & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & \# & \# \\ \# & \# & 0 & 0 \end{bmatrix}$
$1^3 2$-1	$1^3 2$-2	$1^3 2$-3	$1^3 2$-4	$1^3 2$-5
$\begin{bmatrix} \# & \# & \# & 0 \\ \# & 0 & \# & \# \\ \# & \# & 0 & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & \# & 0 \\ \# & \# & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & \# & 0 \\ \# & \# & 0 & 0 \\ \# & \# & \# & \# \end{bmatrix}$		
$1^3 2$-6	$1^3 3$-1	$1^3 3$-2		
$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & \# & \# & \# \\ \# & \# & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & \# & \# & \# \\ \# & \# & \# & 0 \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & \# & \# & \# \\ 0 & 0 & \# & \# \end{bmatrix}$	$\begin{bmatrix} \# & \# & 0 & 0 \\ 0 & 0 & \# & \# \\ \# & \# & \# & \# \end{bmatrix}$	
$1^2 2^2$-1	$1^2 2^2$-2	$1^2 2^2$-3	$1^2 2^2$-4	
$\begin{bmatrix} \# & \# & 0 & 0 \\ \# & \# & \# & 0 \\ \# & \# & \# & \# \end{bmatrix}$				
$1^2 23$-1				

where v_i is the eigenvalue of $\mathbf{A}^T\mathbf{A}$ or $\mathbf{A}\mathbf{A}^T$ or from Eq. (11), $\mathbf{B}\mathbf{B}^T$.

Note that the square root of the product of eigenvalues of $\mathbf{B}\mathbf{B}^T$ is the product of eigenvalues of **B**. In addition, since matrix **B** is a lower triangular matrix, the product of eigenvalues of **B** is the product of its diagonal elements. Hence, FC can be rewritten as

$$\mathrm{FC} = \mu^n \left(\prod_{i=1}^{n} b_{ii} \right)^{-1} \tag{15}$$

The force capacity, FC, can be used as an index to indicate the ability of a TDM to respond to a given set of input tendon forces. The larger the FC, the more responsive the system is. To achieve a fair comparison between TDMs, μ in Eq. (15) can be determined as the following equation by assuming FC equal to 1:

$$\mu = \left(\prod_{i=1}^{n} b_{ii} \right)^{1/n} \tag{16}$$

Hence, from Eq. (16), the global amplification factor of pulley sizing, μ, can be used as an index to indicate the ability of a TDM to respond to a given set of input tendon forces. The smaller the μ, the more responsive the system is.

KINEMATIC SYNTHESIS

Let the matrices **B** and **P** take the following form:

$$\mathbf{B} = \frac{1}{\mu} \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \tag{17}$$

and

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} & p_{1n+1} \\ p_{21} & p_{22} & \cdots & p_{2n} & p_{2n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} & p_{nn+1} \end{bmatrix} \tag{18}$$

By multiplying the equivalent structure matrix **B** and orientation matrix **P**, from Eq. (4), the structure matrix **A** can be determined. From Eqs. (17) and (18), structure matrix **A** can be written as

$$\mathbf{A} = \mathbf{BP} = \frac{1}{\mu} \begin{bmatrix} b_{11} p_{11} & b_{11} p_{12} & \cdots & b_{11} p_{1n+1} \\ b_{21} p_{11} + b_{22} p_{21} & b_{21} p_{12} + b_{22} p_{22} & \cdots & b_{21} p_{1n+1} + b_{22} p_{2n+1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} p_{11} + \cdots + b_{nn} p_{n1} & b_{n1} p_{12} + \cdots + b_{nn} p_{n2} & \cdots & b_{n1} p_{1n+1} + \cdots + b_{nn} p_{nn+1} \end{bmatrix} \tag{19}$$

Let the structure matrix **A** take the following form,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & a_{1n+1} \\ a_{21} & a_{22} & \cdots & a_{2n} & a_{2n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & a_{nn+1} \end{bmatrix} \tag{20}$$

From Eqs. (19) and (20), we have

$$a_{ij} = \frac{1}{\mu} \sum_{k=1}^{i} b_{ik} p_{kj} \tag{21}$$

Since matrix **B** is a lower triangular matrix, from Eqs. (18) and (19), it can be seen that the locations of equivalent tendon actuators specified in **P** are correspondent to the locations of tendon actuators in **A**. Thus, for 2-DOF TDMs, the g^3-series and g^2s-series structure matrices can only be obtained from the 1^3-series and $1^2 2$-series orientation matrices, respectively. For 3-DOF TDMs, the g^4-series, g^3s-series, g^3e-series, g^2s^2-series, and the g^2se-series structure matrices can only be obtained from the 1^4-series, $1^3 2$-series, $1^3 3$-series, $1^2 2^2$-series, and the $1^2 23$-series orientation matrices.

Since the orientation matrix **P** is a unitary matrix,[14] we have

$$\sum_{j=1}^{n+1} p_{ij}^2 = 1 \quad \text{for } i = 1,2,\ldots,n \tag{22}$$

$$\sum_{j=1}^{n+1} p_{ij} p_{kj} = 0 \quad \text{for } i = 1,2,\ldots,n;$$
$$k = 1,2,\ldots,n;\ i \neq k \qquad (23)$$

Substituting Eq. (8) into Eq. (7) yields

$$\mathbf{BPH}_m = 0 \qquad (24)$$

Equation (24) can be reduced to the following form by Gaussian elimination[14] since equivalent structure matrix **B** is a square matrix. Thus,

$$\mathbf{PH}_m = 0 \qquad (25)$$

Equation (25) is a sufficient condition for a TDM to possess a kinematic isotropic condition between the equivalent tendon-space and the tendon-space. Note that, from Eq. (25), the sum of the elements in each row in the matrix **P** is equal to zero, i.e.,

$$\sum_{j=1}^{n+1} p_{ij} = 0 \quad \text{for } i = 1,2,\ldots,n \qquad (26)$$

Note that for an n-DOF TDM with $n+1$ tendons, the number of equation available in Eq. (22) is n, those in Eq. (23) is $n(n-1)/2$, while those in Eq. (26) is n. From Eq. (21), for a_{ij} to be zero, we have

$$\sum_{k=1}^{i} b_{ik} p_{kj} = 0 \qquad (27)$$

For the a_{ij} preceding the first nonzero element of the jth column of **A**, Eq. (27) is always true, while for the a_{ij} succeeding the first nonzero element of the jth column of **A** to be zero, Eq. (27) is an additional constraint in determining the elements of **P**. From Eq. (27), it can be seen that for a_{ij} to vanish, there must be at least two elements from p_{1j} to p_{ij} not equal to zero. Hence, for an n-DOF TDM with $n+1$ tendons with k zero elements succeeding the first nonzero elements in each column, the total number of equations available in determining the elements of orientation matrix **P** is equal to $n(n+3)/2 + k$. Thus, in general, for the solvability of orientation matrix **P**, the number of unknowns in **P** must be equal to or greater than

Table IV. Admissible orientation matrices with which kinematic isotropic transmission characteristics can be obtained.

(a) 2-DOF TDMs

A	Admissible P
g^3-1	1^3-1 1^3-2
g^3-2	1^3-1
g^2s-1	$1^2 2$-1

(b) 3-DOF TDMs

A	Admissible P
g^4-1	1^4-1 1^4-2 1^4-3 1^4-4 1^4-5 1^4-6 1^4-7 1^4-8 1^4-9 1^4-10
g^4-2	1^4-1 1^4-2 1^4-3 1^4-4 1^4-5 1^4-6 1^4-7
g^4-3	1^4-1 1^4-2 1^4-3
g^4-4	1^4-1 1^4-2 1^4-3
g^4-5	1^4-1
g^3s-1	$1^3 2$-1 $1^3 2$-2 $1^3 2$-3 $1^3 2$-4 $1^3 2$-5 $1^3 2$-6
g^3s-2	$1^3 2$-1 $1^3 2$-3 $1^3 2$-4
g^3s-3	$1^3 2$-1 $1^3 2$-2 $1^3 2$-3 $1^3 2$-4
g^3s-4	$1^3 2$-1
g^3s-5	$1^3 2$-1
g^3s-6	$1^3 2$-1
g^3e-1	$1^3 3$-1 $1^3 3$-2
g^3e-2	$1^3 3$-1
g^2s^2-1	$1^2 2^2$-1 $1^2 2^2$-2 $1^2 2^2$-3 $1^2 2^2$-4
g^2s^2-2	$1^2 2^2$-1
g^2s^2-3	$1^2 2^2$-1
g^2se-1	$1^2 23$-1

$n(n+3)/2+k$. Table IV shows the admissible orientation matrices of 2-DOF TDMs with three tendons and 3-DOF TDMs with four tendons. Note that orientation matrices 1^4-10, $1^2 2^2$-3, and $1^2 2^2$-4 are eligible since one of the dot product equations between row vectors is zero by default.

EXAMPLE

In what follows, we shall use a 2-DOF TDM with three tendons as an illustrative example. For the planar TDM shown in Figure 3, the product of the Jacobian matrix can be written as

$$\mathbf{J}^T\mathbf{J} = \begin{bmatrix} a_1{}^2 + a_2{}^2 + 2a_1a_2c_2 & a_2{}^2 + a_1a_2c_2 \\ a_2{}^2 + a_1a_2c_2 & a_2{}^2 \end{bmatrix} \quad (28)$$

where $c_i = \cos(\theta_i)$, $s_i = \sin(\theta_i)$.

From Eqs. (12), (17), and (28), we have

$$b_{11}{}^2 = a_1{}^2 + a_2{}^2 + 2a_1a_2c_2 \quad (29a)$$

$$b_{11}b_{21} = a_2{}^2 + a_1a_2c_2 \quad (29b)$$

$$b_{21}{}^2 + b_{22}{}^2 = a_2{}^2 \quad (29c)$$

For the case g^3-2 structure matrix is desired, as shown in Table IV, only orientation matrix as 1^3-1 is admissible. Let the orientation matrix **P** take the following form. Hence,

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{bmatrix} \quad (30)$$

In determining the elements in the orientation matrix, Eqs. (22), (23), and (26) lead to the following

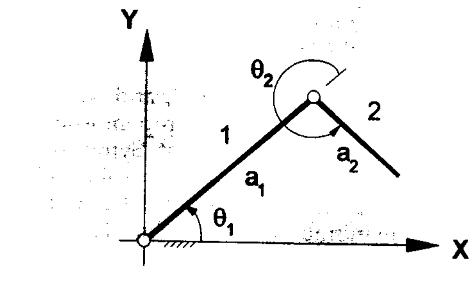

Figure 3. The equivalent open-loop chain of a 2-DOF TDM.

equations,

$$p_{11}^2 + p_{12}^2 + p_{13}^2 = 1 \quad (31a)$$

$$p_{21}^2 + p_{22}^2 + p_{23}^2 = 1 \quad (31b)$$

$$p_{11}p_{21} + p_{12}p_{22} + p_{13}p_{23} = 0 \quad (32)$$

$$p_{11} + p_{12} + p_{13} = 0 \quad (33a)$$

and

$$p_{21} + p_{22} + p_{23} = 0 \quad (33b)$$

From Eq. (19), we have

$$\mathbf{A} = \frac{1}{\mu}\begin{bmatrix} b_{11}p_{11} & b_{11}p_{12} & b_{11}p_{13} \\ b_{21}p_{11} + b_{22}p_{21} & b_{21}p_{12} + b_{22}p_{22} & b_{21}p_{13} + b_{22}p_{23} \end{bmatrix} \quad (34)$$

Note that, for tendon routing g^3-2, there is a zero element succeeding the first nonzero elements of its column vectors. Hence, from Eq. (27), tendon routing g^3-2 can be obtained if one and only one of the following equations is satisfied

$$b_{21}p_{11} + b_{22}p_{21} = 0 \quad (35a)$$

or

$$b_{21}p_{12} + b_{22}p_{22} = 0 \quad (35b)$$

or

$$b_{21}p_{13} + b_{22}p_{23} = 0 \quad (35c)$$

Let the link lengths of the TDM be $a_1 = 10$ cm, $a_2 = 7.07$ cm. Assuming the TDM is positioned at $\theta_1 = 45°$ and $\theta_2 = 60°$, by solving Eqs. (29a–c), we have

$$\mathbf{B} = \frac{1}{\mu}\begin{bmatrix} 14.856 & 0 \\ 5.745 & 4.122 \end{bmatrix} \text{(cm)} \quad (36)$$

Let Eq. (35b) be chosen as the additional constraint equation. By solving Eqs. (31)–(33), and (35b), the elements in orientation matrix **P** can be determined as

$$\mathbf{P} = \begin{bmatrix} 0.813 & -0.476 & -0.337 \\ 0.080 & 0.663 & -0.744 \end{bmatrix} \quad (37)$$

By substituting Eqs. (36) and (37) into Eq. (34) and choosing μ as 5, we have

$$\mathbf{A} = \begin{bmatrix} 2.414 & -1.414 & -1 \\ 1 & 0 & -1 \end{bmatrix} \text{(cm)} \quad (38)$$

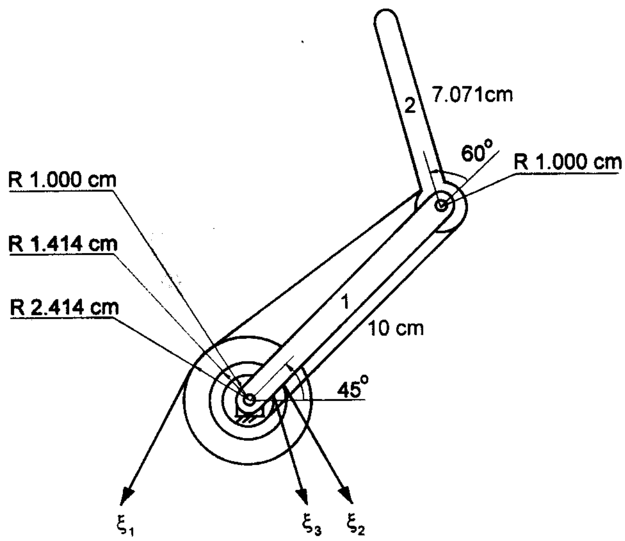

Figure 4. Tendon routings of the g^3-2 TDM.

Figure 4 shows the tendon routings of the example 2-DOF g^3-2 TDM. Note that the positive constant μ in Eq. (34) can be treated as a proportional constant which will not change the isotropic transmission characteristics but will change the resulting tensions in tendons.

SUMMARY

A systematic methodology is developed for the topological synthesis of TDMs. An equivalent tendon space is introduced to simplify the complex transformation between the $(n + 1)$-dimensional tendon space and n-dimensional joint-space. An orientation matrix and an equivalent structure matrix are introduced for the two-step transformation. Admissible orientation matrices are enumerated in a systematic manner. Design equations for the manipulator to possess kinematic isotropic transmission characteristics are derived. It is shown that the equivalent structure matrix can be uniquely determined once the posture of the manipulator is specified. It is also shown that the orientation matrix can be systematically determined from the choice of additional constraint equation. It is believed that this approach grants the designer more freedom and physical insight in determining the tendon routing and pulley sizes than the one-step structure matrix approach proposed by Lee and Tsai[10] and Ou and Tsai.[11]

APPENDIX

The null-space of **A** can be written as

$$\mathbf{H}_A = \left[-D_1, D_2, \ldots, (-1)^i D_1, \ldots, (-1)^{n+1} D_{n+1}\right]^T \quad \text{(A1)}$$

where D_i is the determinant of the matrix formed by deleting the ith column of **A**. From Eqs. (8), (17), and (18), D_i can be written as

$$D_i = \frac{1}{\mu^n} \det(\mathbf{B}) \cdot D_i^* \quad \text{(A2)}$$

where D_i^* is the determinant of the matrix formed by deleting the ith column of **P**. Thus,

$$\mathbf{H}_A = \frac{1}{\mu^n} \det(\mathbf{B}) \cdot \mathbf{H}_p \quad \text{(A3)}$$

Since the determinant of **B** is positive and μ is a positive constant, the null space of $\mathbf{P}, \mathbf{H}_p$ have the same characteristics as $\mathbf{H}_A$, i.e., also of the same sign and nonzero.

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