On the Embedded Kinematic Fractionation of Epicyclic Gear Trains

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In this paper, the concept of kinematic fractionation is introduced for epicyclic gear trains (EGTs) which contains structural fractionation as a degenerate case. With the concept of kinematic fractionation, kinematically independent group(s) embedded in an EGT can be identified. A composition list, which depicts the links and the link connection in the associated group, is used to determine the type of fractionation. It is found that most previously enumerated structurally non-fractionated EGTs in the literature are kinematically fractionated. It is shown that a structurally non-fractionated EGT may be kinematically fractionated while a structurally fractionated EGT must be kinematically fractionated. The concept of kinematic fractionation can lead to efficient topological analysis of EGTs with physical comprehension. [S1050-0472(00)01604-4]

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1 Introduction

The concept of decomposition has been used widely to simplify the complex design processes and to constitute the foundation of modular design. Traditionally, the fractionation of a geared mechanism is recognized from the existence of cut links [1]. However, this approach extremely limits the definition of fractionation since only structural properties are taken into account. In the kinematic analysis of geared mechanisms, it is common that some kinematic equations have to be grouped to obtain a common variable(s) which carry the given input information to the output link. The grouping of equations implies that geared mechanisms may be decomposed according to, not only its physical structure, but the kinematic properties as well.

There are numerous papers and articles on the topological synthesis of geared mechanisms. Applying Boolean algebra, Freudenstein [2] enumerated one-degree-of-freedom (dof) EGTs with up to five links. Tsai [3] presented a generic approach to find one-dof EGTs with up to six links. Tsai and Lin [1] applied a modified generic approach to find structurally non-fractionated two-dof EGTs with up to seven links. With the admissible graph method, Hsu [4] constructed an atlas of non-fractionated EGTs with up to three-dof and four gear pairs. However, in the above results, EGTs were regarded as non-fractionated only from the structural point of view since no cut link can be identified. The embedded kinematic fractionation inside an EGT was not investigated.

In the design decomposition analysis, Wagner and Papalambros [5] used the functional dependence table to show the relation among design variables and to find potential grouping of the objective functions. Kusiak and Larson [6] used the constraint-parameter incidence matrix to decompose the constraints in a design problem. However, by applying the above methods, the EGTs can only be decomposed according to the structural constraints. Hence, a tool for unraveling the embedded grouping of kinematic relations is needed.

In this paper, the traditional definition of fractionation is reviewed and an extensive definition of fractionation is introduced by investigating the embedded kinematic relations in an EGT. An approach based on the generalized vertex-vertex adjacency matrix is developed to find the composition list(s) of an EGT, which contains the links and the link connection in a kinematically independent group. From the composition list, the type of fractionation can be determined. It is shown that a structurally non-fractionated EGT may be kinematically fractionated while a structurally fractionated EGT must be kinematically fractionated. It is found that most 1-dof up to 6-link and 2-dof up to 7-link non-fractionated EGTs enumerated by Freudenstein [2], Tsai [3], Hsu [4], and Tsai and Lin [1] can be further fractionated via the concept of kinematic fractionation. With the concept of kinematic fractionation, the kinematically independent group(s) is identified and the kinematic interactions inside an EGT are exposed explicitly. According to the revealed kinematic insight, it is shown that topological analysis of EGTs can be guided with physical comprehension and completed in an efficient manner.

2 Structural Fractionation

In a graph representation of an EGT, vertices denote links and the joint between links are denoted by edges. A thin edge represents a revolute joint and a heavy edge represents a gear pair. Also, each thin edge is labeled according to its axis position. The following definitions are reviewed for the discussion on fractionations:

a. pseudo-isomorphism

The sub-graph of an EGT obtained by deleting all heavy edges forms a tree [2]. In particular, the connection of a sub-tree composed of vertices having the same label can be rearranged without changing the kinematic characteristics of the corresponding EGT. The operation of coaxial rearrangement is called vertex selection. Two graphs are said to be pseudo-isomorphic if they become isomorphic after one or more vertex selections.

b. articulated kinematic chain

An articulation point is defined as a vertex in a graph whose removal results in an increase of the number of components. A kinematic chain in which sub-graphs are connected by one or more articulation points, is said to be an articulated kinematic chain.

c. cut link

The link corresponding to an articulation point in a graph is called a cut link since the associated kinematic chain can be cut into sub-chains at the cut link.

The concept of structural fractionation regards an EGT as fractionated if cut link(s) exists in the gear train. Accordingly, an EGT can be fractionated through structural fractionation if at least one articulation point can be found in its graph representation or in the

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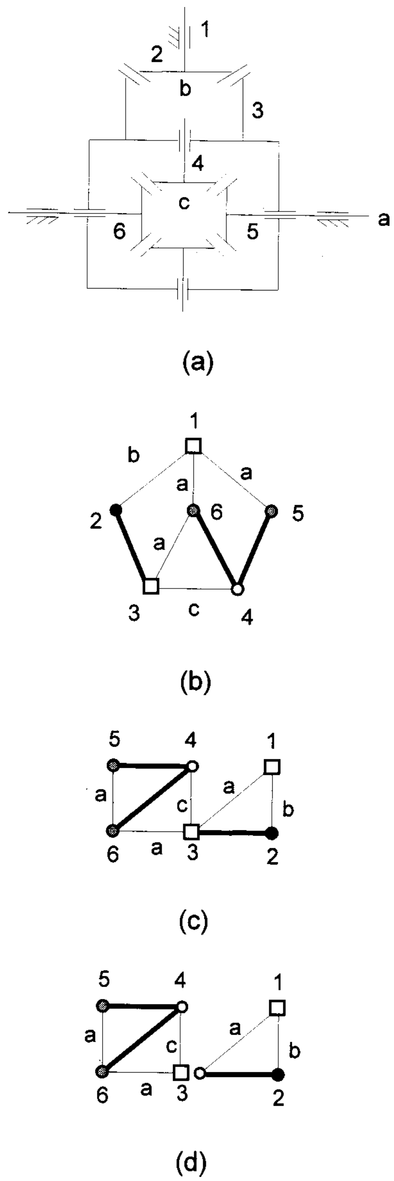


Fig. 1 A differential mechanism; (*a*) The functional representation; (*b*) The graph representation; (*c*) A pseudo-isomorphic graph

derived pseudo-isomorphic graphs. Separating from the cut link, an EGT can be divided into groups which can be viewed as submechanisms. Figure 1(*a*) shows the functional schematic of a differential mechanism and Fig. 1(*b*) is the associated graph representation after deleting redundant links and joints. In Fig. 1(*b*), the square vertices stand for the carriers, the dark vertex for the input link and the shaded vertices for output links. By vertex selections, one of its pseudo-isomorphic graphs is shown in Fig. 1(*c*) in which link 3 is identified as a cut link. Separating the cut link, the differential mechanism is fractionated into two groups as shown in Fig. 1(*d*). Hence, the differential mechanism is a structurally fractionated EGT.

3 Kinematic Fractionation

3.1 The Concept of Kinematic Fractionation. Figure 2(*a*) shows the 1400-1-4 geared kinematic chain [2]. Since no articulation point can be found in all its pseudo-isomorphic graphs, the given kinematic chain is structurally non-fractionated. For the

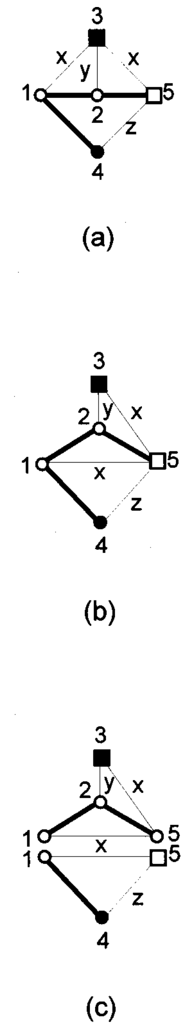


Fig. 2 A 1-dof 5-link EGT (1400-1-4 graph); (*a*) The graph representation; (*b*) A pseudo-isomorphic graph; (*c*) The fractionated groups

geared kinematic chain, the fundamental circuits are (1,2)(3), (5,2)(3) and (1,4)(5) and the associated fundamental circuit equations [7] can be written as

$$\theta_{13}=e_{21}\cdot\theta_{23} \tag{1a}$$

$$\theta_{53}=e_{25}\cdot\theta_{23} \tag{1b}$$

$$\theta_{15}=e_{41}\cdot\theta_{45} \tag{1c}$$

where θ_{ij} is the relative angular displacement between links i and j, and e_{xy} is the ratio of the number of teeth in gear x to that in gear y.

Suppose that links 4, 3 and 5 are selected as input, output, and ground links, respectively. The output angular displacement θ_{35} is to be expressed in terms of the input angular displacement θ_{45}. However, the desired kinematic relation does not show up explicitly. To express θ_{35} in terms of θ_{45}, we need an additional equation that represents the relation among the coaxial links 1, 3 and 5:

$$\theta_{13}=\theta_{15}-\theta_{35} \tag{2}$$

By taking vertex selection among links 1, 3 and 5, a pseudo-isomorphic graph can be obtained as shown in Fig. 2(*b*). In Fig. 2(*b*), a thin edge between vertices 1 and 5 is formed to replace the thin edge between vertices 1 and 3. From Fig. 2(*b*), it can be seen that the angular displacement θ_{15} is influenced by the input angular displacement θ_{45} through fundamental circuit (1,4)(5). In turn,

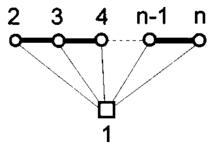


Fig. 3 A typical group

θ_{15} influences the fundamental circuits (1,2)(3), (5,2)(3) simultaneously since vertices 1 and 5 are at the two ends of the heavy-edged path 1-2-5. The pseudo-isomorphic graph shown in Fig. 2(*b*) thus, provides a clearer picture of the connection among the three fundamental circuits. It can be seen that the three fundamental circuits are divided into two groups, one is (1,4)(5) and the other consists of (1,2)(3) and (5,2)(3), with θ_{15} being the common variable between the sets of equations representing the two groups. Corresponding to the common variable, vertices 1 and 5, and their connecting thin edge are considered a common linkage between the two groups. Separating the graph from the common linkage results in the fractionation shown in Fig. 2(*c*). In Fig. 2(*c*), the upper group consists of the two coupled fundamental circuits (1,2)(3), (5,2)(3), and the lower group consists of the fundamental circuit (1,4)(5).

From the lower group, the input information is transformed in terms of θ_{15} by Eq. (1*c*) and is transmitted to the upper group through the common linkage in between. From Eqs. (1*a*) and (1*b*), the output angular displacement θ_{35} can be written as

$$\theta_{35}=-\theta_{53}=-e_{25}\cdot\theta_{23}=-\frac{e_{25}}{e_{21}}\cdot\theta_{13} \tag{3}$$

Substituting Eq. (2) into Eq. (3), yields

$$\theta_{35}=\frac{e_{25}}{e_{25}-e_{21}}\cdot\theta_{15} \tag{4}$$

By substituting Eq. (1*c*) into Eq. (4), we have

$$\theta_{35}=\frac{e_{25}\cdot e_{41}}{e_{25}-e_{21}}\cdot\theta_{45} \tag{5}$$

From the above derivation, it can be seen that the group associated with Eq. (1*c*) is used to transform the angular displacement of the input link while the group associated with Eqs. (1*a*) and (1*b*) is used to generate the angular displacement of the output link. The common thin edge between the two groups receives the input information and carries into the group containing the output link. The derivation shows that proper grouping of fundamental circuit equations can lead to the fractionation of embedded kinematic relations.

From the observation of Fig. 2(*c*), it is found that there is a common transfer vertex for gear pairs in each fractionated group. The observation implies that the separation of transfer vertices leads to the fractionation of coupled kinematic relations. In each fractionated group, fundamental circuits are serially connected as the n-link case shown in Fig. 3. The geared path conveys the kinematic information to each fundamental circuit in the same group in sequence. In view of the distinct power transmission path, each group can be regarded as a sub-mechanism in the EGT.

3.2 Kinematic Fractionation vs. Structural Fractionation. The concept of structural fractionation uses the existence of cut link(s) as the foundation of fractionation while the concept of kinematic fractionation fractionates an EGT according to the embedded kinematic relations. However, similar characteristics still can be found from the two types of fractionation:

1 From the concept of structural fractionation, the removal of the articulation point increases the number of components in a geared kinematic chain. From the concept of kinematic fractionation, removal of the vertices and edges on the common linkage can also increase the number of components of a geared kinematic chain.

2 Separation of a cut link results in structurally independent sub-mechanisms while the separation of the common linkage leads to sub-mechanisms which are kinematically independent to others.

It can be found that by regarding a cut link as the degenerate case of a common linkage, a structurally fractionated EGT can also possess the characteristics of kinematically fractionation. In contrast, a kinematically fractionated EGT is not necessarily structurally fractionated, such as the 1400-1-4 graph in Fig. 2(*a*). Hence, the concept of kinematic fractionation contains the concept of structural fractionation as a sub-case.

4 Determination of the Type of Fractionation

By modifying the link adjacency matrix [8], the generalized vertex-vertex adjacency matrix is introduced to express the relations in a geared kinematic chain and can be defined as:

$$\mathbf{V}=[\mathrm{v}_{ij}] \tag{6}$$

where $\mathrm{v}_{ij}=0$ if there is no joint between the links corresponding to vertices i and j, $\mathrm{v}_{ij}=\mathrm{G}$ if the links corresponding to vertices i and j form a geared pair, $\mathrm{v}_{ij}=$axis label if vertices i and j represent coaxial links, and $\mathrm{v}_{ii}=1$.

Such as the 1400-1-4 graph shown in Fig. 2(*a*), the generalized vertex-vertex adjacency matrix can be expressed as

$$\mathbf{V}=\begin{bmatrix} 1 & \mathrm{G} & \mathrm{x} & \mathrm{G} & \mathrm{x} \\ \mathrm{G} & 1 & \mathrm{y} & 0 & \mathrm{G} \\ \mathrm{x} & \mathrm{y} & 1 & 0 & \mathrm{x} \\ \mathrm{G} & 0 & 0 & 1 & \mathrm{z} \\ \mathrm{x} & \mathrm{G} & \mathrm{x} & \mathrm{z} & 1 \end{bmatrix} \tag{7}$$

For a given vertex-vertex adjacency matrix, the following properties of the corresponding gear train are revealed:

1 A link can be recognized as the carrier associated with a gear pair if more than two axis labels can be found in the corresponding row.
2 The relation between a gear pair and the identified carriers can be determined according to the following rule:

For $\mathrm{V}_{ij}=\mathrm{G}$, link k is the carrier between gears i and j if V_{ik} and V_{jk} have different axis labels since a carrier must carry a pair of gears in different axes.

Based on the above properties of vertex-vertex adjacency matrix, a procedure to find the kinematically independent group(s) in an EGT is developed as follows:

Step 1: Establish the generalized vertex-vertex adjacency matrix
Step 2: Identify the carrier(s) and the associated gear pairs

By examining all the G elements in upper triangle of a vertex-vertex adjacency matrix, the carrier(s) and the associated gear pairs can be found.

For the 1400-1-4 graph, link 3 and link 5 are recognized as carriers. From Eq. (7), the G elements are found in $\mathrm{V}_{12}, \mathrm{V}_{14}, \mathrm{V}_{25}$. For V_{12}, it is found that $\mathrm{V}_{13}=\mathrm{x}$ and $\mathrm{V}_{23}=\mathrm{y}$. Hence, gear links 1 and 2 are carried by link 3, and the relation is denoted as (1,2)(3). Similarly, gear links 1 and 4 are found being carried by link 5, and gear links 2 and 5 by link 3. The gear pairs and associated carriers are denoted as (1,4)(5) and (2,5)(3), respectively.

Step 3: Construct the composition list

The composition of a group can be found by collecting the connected gear pairs carried by an identical carrier. The composition of a group is denoted as a composition list in the following form:

$$[\text{carrier;(gear pair),(gear pair)}, \cdots] \tag{8}$$

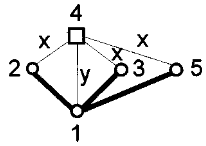


Fig. 4 The graph representation of the 2210-1-4a graph

For the above example, two composition lists can be found as

$$[3;(1,2),(2,5)] \tag{9a}$$

$$[5;(1,4)] \tag{9b}$$

A composition list reveals not only the links in a group but the connections among links are shown as well. Note that the grouping of the links in Eqs. (9*a*) and (9*b*) is identical to the graph representation in Fig. (2*c*).

From the above steps, the kinematically independent groups in an EGT can be identified. The composition lists provide the following information:

1 The number of composition lists is equal to the number of kinematically independent groups in an EGT.
2 The common member(s) of two composition lists represents the link(s) on the common linkage of the associated groups.

According to the above two properties, the obtained composition lists can be used to determine the type of fractionation:
Type 1: An EGT is kinematically non-fractionated if only one composition list is obtained.

Considering the 2210-1-4a graph [2] in Fig. 4, the generalized vertex-vertex adjacency matrix can be expressed as:

$$\mathbf{V}=\begin{bmatrix} 1 & G & G & y & G \\ G & 1 & x & x & x \\ G & x & 1 & x & x \\ y & x & x & 1 & x \\ G & x & x & x & 1 \end{bmatrix} \tag{10}$$

The gear pairs and the associated carriers can be found as (1,2)(4), (1,3)(4) and (1,5)(4), and the composition list can be expressed as

$$[4;(1,2),(1,3),(1,5)] \tag{11}$$

Since only one composition list is obtained, the 2210-1-4a graph is kinematically non-fractionated.
Type 2: An EGT is both structurally and kinematically fractionated if more than one composition lists can be obtained and the number of common members of the composition lists is equal to one.

For the differential mechanism in Fig. 1(*a*), the generalized vertex-vertex adjacency matrix can be expressed as

$$\mathbf{V}=\begin{bmatrix} 0 & b & a & 0 & a & a \\ b & 0 & G & 0 & 0 & 0 \\ a & G & 0 & c & a & a \\ 0 & 0 & c & 0 & G & G \\ a & 0 & a & G & 0 & a \\ a & 0 & a & G & a & 0 \end{bmatrix} \tag{12}$$

From Eq. (12), links 1 and 3 are recognized as the transfer vertices and the composition lists are determined as

$$[1;(2,3)] \tag{13a}$$

$$[3;(4,5),(4,6)] \tag{13b}$$

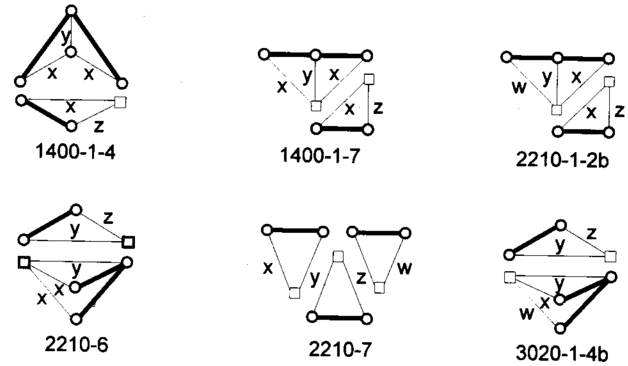


Fig. 5 1-dof 5-link kinematically fractionated GKCs

Table 1 Fractionated results of 1-dof up to 6-link and 2-dof up to 7-link EGTs

# of dof	# of links	EGTs	SNKF EGTs
1	5	13 (Freudenstein, 1971)	6
	6	81 (Tsai, 1987; Hsu, 1993)	54
2	6	3 (Lin and Tsai, 1989)	3
	7	50 (Lin and Tsai, 1989)	50

From Eqs. (13*a*) and (13*b*), it can be found that links (1,2,3) form a group and links (3,4,5,6) form another group. The two composition lists have only one common member, link 3, which represents the cut link between the two groups. Hence, the differential mechanism is both structurally and kinematically fractionated. Note that a cut link represents a degenerate case of the common linkage.
Type 3: An EGT is structurally non-fractionated but is kinematically fractionated (SNKF) if more than one composition lists can be obtained and the number of common members of the composition lists is greater than one.

For the case of 1400-1-4 graph, Eqs. (9*a*) and (9*b*) represents the obtained two composition lists in which two common members, links 1 and 5, are found. Hence, the 1400-1-4 graph is structurally non-fractionated but is kinematically fractionated.

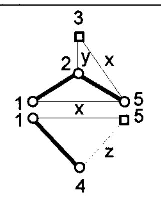
Applying the above procedure to the existing atlas of 1-dof structurally non-fractionated EGTs [2,3,4], it can be found that for five-link case, there are originally thirteen non-fractionated EGTs and six of them can be further fractionated via the concept of kinematic fractionation as shown in Fig. 5. As for six-link case, fifty-four of the eighty-one non-fractionated EGTs are determined as kinematically fractionated. Table 1 shows the fractionated results of the 1-dof up to 6-link and 2-dof up to 7-link structurally non-fractionated EGTs. From Table 1, it can be seen that all two-dof structurally non-fractionated EGTs are kinematically fractionated.

5 Application

Topological analysis is the procedure to determine all admissible ways of assigning ground, input and output links (G/I/O) in a given kinematic chain [9]. Based on the following topological requirements of the desired mechanism.

1 Input and output must be connected to the fixed link via revolute joints,
2 The gear train must not contain any links which do not have any effect on the overall gear ratio,

Table 2 G/I/O assignment of the 1400-1-4 graph

1400-1-4	(G; I/O)	redundant links
	(1; 3/ 5)	4
	(3; 1/ 2)	4, 5
	(3; 1/ 5)	4
	(3; 2/ 5)	1, 4
	(5; 1/ 3)	4
	(5; 1/ 4)	2, 3
	(5; 3/ 4)	none

Olson et al. [9] introduced an inspection approach for the topological analysis by using the coincident-joint graph to determine all possible ways to assign G/I/O. From these possible arrangements, admissible configurations are determined by eliminating those configurations with superfluous (redundant) links. Although this approach is quite straightforward but is exhaustive in determining all assignments. Table 2 shows the possible ways [9] to assign G/I/O for the 1400-1-4 graph. It can be found that there are seven possible ways to assign G/I/O but only one set of G/I/O assignment is valid.

With the concept of kinematic fractionation, a supplementary limitation is provided in determining possible G/I/O assignment. Since each fractionated group is kinematically independent the G/I/O must be distributed in these groups. For a group without functionary links, the group must be redundant in the whole mechanism. For the same 1400-1-4 graph, two groups are obtained via kinematic fractionation as shown in Fig. 2(*c*). To locate the input and output links adjacent to the ground, a thin-edged path connecting three vertices is required. Also, the thin-edged path must traverse the two groups to avoid groups being redundant. From Fig. 2(*b*), 4-5-3 is recognized as the only one possible thin-edged path to locate G/I/O. On the thin-edged path, the ground can be assigned to vertices with degree larger than or equal to two. Hence, on the path 4-5-3, vertex 5 is found to be the only feasible location for ground and vertices 4 and 3 is assigned for input and output links. Note that, the distinction between the input and output links is unnecessary since the power flow path is kept the same but merely with inverse direction. In the above illustration, only one set of G/I/O is tried for the 1400-1-4 graph and the correct configuration is obtained directly.

From the above analyzing process, it can be seen that the concept of kinematic fractionation provides a clear physical understanding about the kinematic relations between groups and leads to a rational way to determine the G/I/O assignment. Comparing to the result of Olson et al. [9], it can be seen that the topological analysis with the add of kinematic fractionating obtains the same result but is completed in a much more efficient manner since unnecessary G/I/O assignments can be avoided.

6 Conclusion

The concept of kinematic fractionation has been introduced to disclose the kinematic interactions within an EGT. The basis to fractionate EGTs is extended to take kinematic properties into account. A systematic procedure is presented to find the kinematically independent group(s) embedded in an EGT. It is shown that the kinematically fractionated EGTs may be structurally fractionated while the structurally fractionated EGTs must be kinematically fractionated. Based on the concept of kinematic fractionation, an efficient method for topological analysis of EGTs that provides clear physical understanding has been developed.

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