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Design of Perfectly Statically Balanced One-DOF Planar Linkages With Revolute Joints Only

A systematic methodology for the design of a statically balanced, single degree-of-freedom planar linkage is presented. This design methodology is based on the concept of conservation of potential energy, formulated by the use of complex number notations as link vectors of the linkage. By incorporating the loop closure equations and the kinematic constraints, the gravitational potential energy of the system can be formulated as the function of the vectors of all ground-adjacent links. The balance of the gravitational potential energy of the system is then accomplished by the elastic potential energy of a zero free-length spring on each ground-adjacent link of the linkage. As a result, spring constants and installation configurations of the ground-attached springs are obtained. Since the variation in the gravitational potential energy of the linkage at all configurations can be fully compensated by that of the elastic potential energy of the ground-attached springs, this methodology provides an exact solution for the design of a general spring balancing mechanism without auxiliary parallel links. Illustrations of the methodology are successfully demonstrated by the spring balancing designs of a general Stephenson-III type six-bar linkage and a Watt-I type six-bar linkage with parallel motion. [DOI: 10.1115/1.3087548]

Keywords: *static balance, zero free-length spring, planar linkages, potential energy*

1 Introduction

Statically balanced mechanisms are capable of self-sustaining the system payloads and require little external actuating force to move objects in a vertical plane motion. Numerous applications of such mechanisms had been employed in the designs of angle-poised lamps [1], surgical light apparatus [2–4], weight-lifting crane [5,6], robot support arm [7,8], etc. Literature showed that the static balance of a mechanism could be achieved by the counterweight method, the joint friction method, the springs and auxiliary parallel links method, etc. Among them, the counterweight method needs to add additional masses to the system and it may undermine the strength of links as the payload increases, while the joint friction method facilitates abrasive force of a joint to counterbalance the gravitational forces of the system and it may cause the unanticipated failure of the system as the joint friction decays. On the other hand, the springs and parallel auxiliary links method was used as an alternative for the design of a statically balanced mechanism. By tracking the center of mass of a mechanism at various configurations, many spring balancing mechanisms (SBMs) are successfully implemented by this method [9–11]. Over the years, the "perfect" spring balances of a single link [1,12,13], single degree-of-freedom (DOF) planar linkages [14], and spatial multi-DOF mechanisms [11,15,16] were investigated. The "perfect equilibration" refers to a system that is in static equilibrium at its all configurations where the effect of gravity is fully eliminated from the system of interest [13,17]. Most of the design methodologies use auxiliary parallel links, forming parallelograms, along with a number of springs attached between the linkage and the auxiliary links to compensate for the variation in the gravitational forces of a linkage. However, the use of auxiliary parallel links induces extra loadings to the system and complicates the arrangement of links and springs to avoid motion interferences. Such a methodology also increases system inertia and, thus, sacrifices its most advantaged aspect in comparison with the counterweight method. Few studies had been focused on the design of a perfectly statically balanced mechanism using only linear springs. Wongratanaphisan and Cole [18] analyzed a four-bar linkage suspended by linear springs without auxiliary parallel links. They studied the sensitivity of the potential energy of the system and showed that a four-bar linkage can be perfectly balanced only when certain symmetric geometry and mass conditions were satisfied. Shieh et al. [19] proposed a methodology for the design of a spring balancing general four-bar linkage without auxiliary parallel links. However, the static balance of SBMs of higher links was not investigated.

In this paper, a systematic methodology based on the concept of conservation of potential energy is proposed for the design of a statically balanced, one-DOF, all-revolute, planar linkage without the use of auxiliary parallel links. In the methodology, the gravitational forces of the feasible linkages with any geometry or mass properties can be perfectly balanced. By equating the variations in the gravitational potential energy of the linkage and the elastic potential energy of the springs attached to the system, necessary and sufficient conditions for the static balance of one-DOF, all-revolute, planar linkages are obtained, and the spring constant and installation configuration of each spring are determined. Based on the design methodology and the use of graph representations, admissible one-DOF SBMs of up to eight links are enumerated. Designs of the SBMs of a general Stephenson-III type six-bar linkage and a Watt-I type six-bar linkage with parallel motion are demonstrated for the achievement of the design methodology.

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Contributed by the Mechanism and Robotics Committee of ASME for publication in the JOURNAL OF MECHANICAL DESIGN. Manuscript received May 30, 2008; final manuscript received January 6, 2009; published online April 3, 2009. Review conducted by Pierre M. Larochelle, P.E. Paper presented at the ASME 2008 Design Engineering Technical Conferences and Computers and Information in Engineering Conference (DETC2008), Brooklyn, NY, August 3–6, 2008.

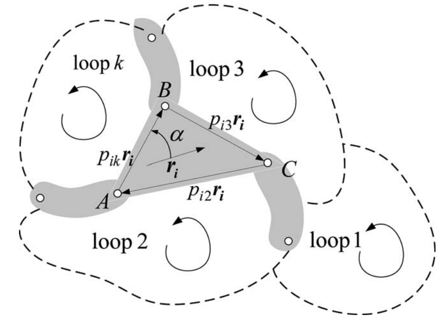


Fig. 1 Schematic of independent loops of a one-DOF planar linkage

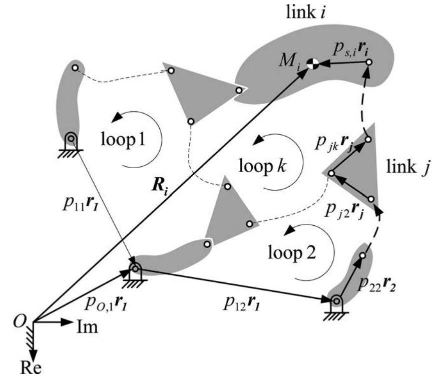


Fig. 2 Path of a distal mass center from the fixed frame

2 Design Concept

The general form of the gravitational potential energy of a one-DOF, all-revolute, planar linkage can be formulated by the link vectors of the linkage. A link vector is defined as any arbitrary nonzero vector fixed on the considered link. Because the link vectors of a planar linkage are constrained by their associated loop closure equations and the kinematic constraints, the number of the linearly independent link vectors can be obtained by subtracting the number of the loop closure equations along with the kinematic constraints from the number of links. These linearly independent link vectors are then referred as the base vectors, or the basis [20] spanning the space of the gravitational potential energy of the system. Although the selection of the base vectors is not unique, for simplicity and without loss of generality, vectors representing the ground and the ground-adjacent links are chosen as the base vectors. With such an arrangement, the overall gravitational potential energy of the linkage is equivalent to the sum of the gravitational potential energy expressed in terms of ground and the ground-adjacent links. Since the variation in the gravitational potential energy of any ground-adjacent link can be fully compensated by the elastic potential energy of a zero free-length, extensional, linear spring fitted between ground and the ground-adjacent link, the static balance of the linkage can be easily accomplished. Since the number of ground and the ground-adjacent links has to be equal to that of the base vectors, certain one-DOF planar linkages are not admissible for the design of the SBMs with springs alone. Feasible one-DOF SBMs are systematically enumerated by assigning different ground location to a kinematic chain, and the static balance of the linkage is accomplished by fitting a ground-attached spring with specified design parameters to its corresponding link.

3 Gravitational Potential Energy of General Planar linkages

3.1 Formulation. The gravitational potential energy of a link of a planar linkage can be represented by the scalar product of the position vector of the mass center and the vector of the gravitational force of the link. For simplicity and without suffering the complicated algebraic operations of trigonometric functions, complex number notations are used for vectors in the derivations for both elastic and gravitational potential energies of the system. Consider link i as any link of a planar linkage, as shown in Fig. 1, where $\boldsymbol{r_i}$ is an arbitrary vector fixed on link i, and points A, B, and C are fixed on link i. Link vector $\boldsymbol{r_{AB}}$ can be written as

$$\boldsymbol{r_{AB}} = \left(\frac{|\boldsymbol{r_{AB}}|}{|\boldsymbol{r_i}|}e^{i\alpha}\right)\boldsymbol{r_i} = p_{ik}\boldsymbol{r_i} \tag{1}$$

where $|\cdot|$ denotes the magnitude of the vector inside the norm, α is the angle measured from vectors $\boldsymbol{r_i}$ to $\boldsymbol{r_{AB}}$, $(|\boldsymbol{r_{AB}}|/|\boldsymbol{r_i}|)$ is the elongation ratio of vector $\boldsymbol{r_{AB}}$ with respect to vector $\boldsymbol{r_i}$, and p_{ik} is the constant coefficient transforming vector $\boldsymbol{r_i}$ to vector $\boldsymbol{r_{AB}}$ on loop k. Since both vectors $\boldsymbol{r_i}$ and $\boldsymbol{r_{AB}}$ are fixed on link i, coefficient p_{ik} is independent of the configuration of the linkage and, hence, p_{ik} is a constant coefficient expressed in exponential form of a complex number. Note that all vectors hereafter are represented in boldface characters, while constant coefficients are not.

Since the link vector on link i and loop k can be represented by $p_{ik}\boldsymbol{r_i}$, link vectors $\boldsymbol{r_{BC}}$ and $\boldsymbol{r_{CA}}$ can be represented by $p_{i3}\boldsymbol{r_i}$ and $p_{i2}\boldsymbol{r_i}$, respectively. Note that if link i is not on loop k, p_{ik} is equal to zero, e.g., p_{i1} in Fig. 1 is zero. Also note that, throughout the paper, all angles are measured counterclockwise, and the directions of link vectors $p_{i2}\boldsymbol{r_i}$, $p_{i3}\boldsymbol{r_i}$, $p_{ik}\boldsymbol{r_i}$, etc., are in accordance with the direction of their corresponding loop, as illustrated in Fig. 1 unless specified otherwise.

Referring to the general planar linkage in Fig. 2, M_i is the mass center of link i and $\boldsymbol{R_i}$ is the position vector of mass center M_i from point O of the reference coordinate system. The coordinate system of the complex plane is placed with its positive real-axis pointing downward and imaginary-axis rightward. Based on Eq. (1), position vector $\boldsymbol{R_i}$ in Fig. 2 can be expressed as

$$\boldsymbol{R_i} = p_{O,1}\boldsymbol{r}_1 + \left(\sum_j \sum_k p_{jk}\boldsymbol{r_j}\right) + \boldsymbol{s_i} = p_{O,1}\boldsymbol{r}_1 + p_{s,i}\boldsymbol{r_i} + \sum_j \sum_k p_{jk}\boldsymbol{r_j} \tag{2}$$

where j and k are the labels of the link and the loop, respectively, on the path from point O to the mass center of link i, $\boldsymbol{s_i}$ is the position vector of mass center M_i from the preconnected joint of link i, and $p_{s,i}$ is the constant transformation coefficient of vector $\boldsymbol{r_i}$ to vector $\boldsymbol{s_i}$.

Hence, the gravitational potential energy of link i can be expressed as

$$(e_g)_i = -m_i\boldsymbol{g}\cdot\boldsymbol{R_i} = \boldsymbol{g}\cdot\left(-m_i p_{O,1}\boldsymbol{r}_1 - m_i p_{s,i}\boldsymbol{r_i} - m_i\sum_j \sum_k p_{jk}\boldsymbol{r_j}\right) \tag{3}$$

where $\boldsymbol{g}$ represents the vector of the gravitational acceleration, pointing downward. The negative sign in Eq. (3) implies that the maximum gravitational energy of link i is at the highest position of its mass center.

Note that, for a closed-loop, planar linkage, the path selected from the reference coordinate to the mass center M_i is not unique, and thus the representation of Eq. (3) is also not unique. By collecting all constant coefficients for each link vector $\boldsymbol{r_i}$, the overall gravitational potential energy of an n-link planar linkage becomes

$$e_g = \sum_i (e_g)_i = \boldsymbol{g}\cdot\left(\sum_{i=1}^{n} c_i\boldsymbol{r_i}\right) \tag{4}$$

where c_i, expressed in exponential form of complex number, is a constant coefficient constituted by m_i, p_{jk}, and $p_{s,i}$.

Equation (4) indicates that the gravitational potential energy of an n-link planar linkage can be expressed by the inner product of vector $\boldsymbol{g}$ and a vector formed by the linear combination of $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n$. However, since the n-link vectors of a closed-loop planar linkage are not mutually linearly independent, c_i may be different, and a unique expression of c_i is not available. Hence, the number of the linearly independent link vectors used in the formulation of the gravitational potential energy in Eq. (4) has to be identified.

3.2 Kinematic Constraints of a Linkage System. Kinematic constraints of a planar linkage confine the relative motions among links. In general, a linear kinematic constraint equation can be expressed as

$$\sum_{k=1}^{n} q_k \boldsymbol{r}_k = 0 \tag{5}$$

where q_k is the constant coefficient and q_k can be zero, a real or a complex number.

Assume $q_k=0$ for $k \neq i, j$, Eq. (5) is reduced to

$$\boldsymbol{r}_i + \left(\frac{q_j}{q_i}\right)\boldsymbol{r}_j = 0 \tag{6a}$$

or

$$\boldsymbol{r}_i = q\boldsymbol{r}_j = \left(\frac{|\boldsymbol{r}_i|}{|\boldsymbol{r}_j|}e^{i\alpha}\right)\boldsymbol{r}_j \tag{6b}$$

where $q=-(q_j/q_i)$, α is the angle measured from vectors $\boldsymbol{r}_j$ to $\boldsymbol{r}_i$, and $(|\boldsymbol{r}_i|/|\boldsymbol{r}_j|)$ is the elongation ratio of the magnitudes of vector $\boldsymbol{r}_i$ with respect to $\boldsymbol{r}_j$.

The kinematic constraint depicted in Eq. (6) suggests that links i and j have the same angular velocity, i.e., links i and j are always kept in a same relative angle of orientation. If links i and j are adjacent, they are considered rigidly connected since there is no relative motion between the two links. However, if links i and j are not adjacent, the two links are in parallel motion and Eq. (6) is referred to the case where the two links are subjected to a kinematic constraint of parallel motion. A parallel kinematic constraint can be resulted from the special arrangement of the link dimensions, e.g., the lengths of the parallel links of a parallelogram must be equal.

On the other hand, for a multiloop, planar linkage, each independent loop represents a linear kinematic constraint to the motion of the links of the planar linkage. The loop closure equation of loop k can be represented as

$$\sum_{i=1}^{n} p_{ik} \boldsymbol{r}_i = 0 \tag{7}$$

For a one-DOF, n-link, planar linkage, the number of the independent loops is $(n/2-1)$ [21]. Writing Eq. (7) for each independent loop k yields a total number of $(n/2-1)$ linearly independent vector equations with n-link vectors. The number of linearly independent link vectors of the linkage is then obtained by subtracting the number of loop closure equations from that of link vectors as

$$\#\{B\} = n - \left(\frac{n}{2} - 1\right) = \frac{n}{2} + 1 \tag{8}$$

where B is referred to the set of linearly independent link vectors or the base vectors of the linkage, and $\#\{B\}$ represents the number of the base vectors.

A set of base vectors, or a basis, is a spanning set of a vector space, where all vectors in that space can be represented by the linear combination of the base vectors in the set. For example, if $B=\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is a basis of vector space V, any vector $\boldsymbol{v} \in V$ can be expressed uniquely by the linear combination of the basis B as $\boldsymbol{v}=\beta_1\boldsymbol{v}_1+\beta_2\boldsymbol{v}_2+\cdots+\beta_n\boldsymbol{v}_n$, where scalars $\beta_1, \beta_2, \ldots, \beta_n$ are the coordinates of vector space V with respect to the basis B.

Solving the $(n/2-1)$ loop closure equations simultaneously, any link vector $\boldsymbol{r}_i$ of the linkage can be expressed by the linear combination of the base vectors as

$$\boldsymbol{r}_i = \sum_k c'_k \boldsymbol{r}_k, \quad \boldsymbol{r}_k \in B \tag{9}$$

where $\boldsymbol{r}_k$ represents a base vector and c'_k is the constant coefficient obtained from the algebraic manipulations of the $(n/2-1)$ loop closure equations of Eq. (7).

Substituting Eq. (9) into Eq. (4), the gravitational potential energy of a one-DOF, planar linkage can be written in terms of the $(n/2+1)$ base vectors as

$$e_g = \boldsymbol{g}\cdot\left(\sum_{i=1}^{n} c_i \boldsymbol{r}_i\right) = \boldsymbol{g}\cdot\left(\sum_k c''_k \boldsymbol{r}_k\right), \quad \boldsymbol{r}_k \in B \tag{10}$$

Although any $(n/2+1)$ of the n independent vectors of the system can be chosen as the base vectors, practically, it is desirable to select ground vector $\boldsymbol{r}_1$ to be one of the base vectors. Since ground vector is configuration independent, vector $\boldsymbol{r}_1$ is referred as the invariant base vector, while the remaining $(n/2)$ base vectors are referred as the variant base vectors. Hence, Eq. (10) can be further rewritten as

$$e_g = \boldsymbol{g}\cdot\left(\sum_k c''_k \boldsymbol{r}_k\right) + \text{const}, \quad \boldsymbol{r}_k \in B \quad \text{and} \quad \boldsymbol{r}_k \neq \boldsymbol{r}_1 \tag{11}$$

where $\boldsymbol{r}_k$ stands for a variant base vector and the constant term is $\boldsymbol{g}\cdot(c''_1\boldsymbol{r}_1)$.

Furthermore, since the rotation axes of the ground-adjacent links are fixed on ground, if the $(n/2)$ variant base vectors represent the link vectors of the ground-adjacent links, the overall gravitational potential energy of a one-DOF planar linkage in Eq. (11) is equivalent to the sum of the gravitational potential energies of the $(n/2)$ ground-adjacent links. Note that, in the expression of the gravitational potential energy in Eq. (11), the multiplication of the mass and the mass center transformation coefficient for each ground-adjacent link, say, link k, are replaced by c''_k. Equation (11) further implies that the number of the variant base vectors has to be equal to or less than the number of the ground-adjacent links if the gravitational potential energy is to be represented by all base vectors. It is well known that the maximum number of links connected to any link in a one-DOF planar linkage is $(n/2)$. Hence, if the number of the variant base vectors is $(n/2)$, the number of the ground-adjacent links of a linkage has to be $(n/2)$; otherwise the gravitational potential energy cannot be represented by the vectors of the ground-adjacent links alone. However, if the number of the variant base vectors is $(n/2-1)$, the number of the ground-adjacent links of a linkage can be $(n/2-1)$ or $(n/2)$. For example, if a coupler link of a one-DOF, planar linkage is in translational motion, the relative orientation of the link with respect to the ground is always constant, and the vector representing the translational link becomes linearly dependent with respect to that of ground link. Hence, the number of the variant base vectors used in the formulation of the gravitational potential energy can be reduced by 1. As a result, the required minimum number of ground-adjacent links is $(n/2-1)$. In general, the required minimum number of ground-adjacent links for an n-link, one-DOF planar linkage can be obtained by subtracting the number of the additional kinematic constraints from $(n/2)$.

Moreover, as indicated previously, the constant coefficient c_i in Eq. (4) may differ if a distinct path is chosen. However, once the base vectors are selected, the constant coefficient c''_k in Eq. (10) is uniquely determined regardless of the path selection since the base vectors are mutually linearly independent. The constant co-

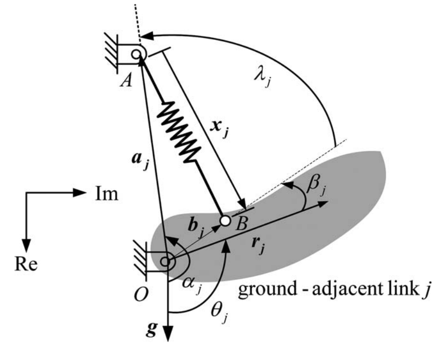


Fig. 3 Installation of a ground-attached spring

efficient c''_k is a function of c_i and c'_k, where c_i contains the mass properties of the linkage as indicated in Eqs. (3) and (4), while, in Eq. (7), c'_k contains only the geometric information of the linkage from the loop closure equations and other kinematic constraints. Note that, through manipulating the mass center locations and the masses of links, it is possible to adjust one or more of the coefficients c''_k to be zero. As a result, the number of linearly independent base vectors for the formulation of the gravitational potential energy can be further reduced. Manipulation on masses and locations of mass centers can be easily achieved by the counterweight method or using extremely lightweight links. Design of a SBM utilizing a hybrid method of both counterweights and springs method is also possible. Such a hybrid method uses less number of springs and, thus, reduces the possibility of sabotage of the integrity of the design due to the possible spring relaxation with time.

4 Elastic Potential Energy of Ground-Attached Springs

If the elastic potential energy of the springs fitted in a one-DOF planar linkage can be represented by the same base vectors used in the formulation of the gravitational potential energy in Eq. (11), static balance of the system becomes possible. Refer to the ground-adjacent link fitted with a spring, as shown in Fig. 3, where link 1 represents ground, link j is the inverted pendulum connected to ground via a fixed pivot, and θ_j is the orientation angle of vector $\boldsymbol{r_j}$ measured with respect to the positive real-axis. By fitting a zero free-length, extensional, linear spring on link j, the elastic potential energy of the ground-adjacent link can be obtained as

$$(e_s)_j = \tfrac{1}{2}K_j|\boldsymbol{x}_j|^2 = \tfrac{1}{2}K_j\boldsymbol{x_j}\cdot\boldsymbol{x_j} \tag{12}$$

where

$$\boldsymbol{x_j} = \boldsymbol{b_j} - \boldsymbol{a_j} \tag{13}$$

and K_j is the spring constant and $\boldsymbol{a_j}$ and $\boldsymbol{b_j}$ are the position vectors of the attachment ends of the spring at pivots A and B, respectively.

Note that a zero free-length spring can be made up with a pretensioned, nonzero free-length spring and possibly with cables and pulleys [14,15]. The magnitude of preload is arranged to be equal to the product of the spring constant and the free length of the spring at the configuration where the two ends of the zero free-length spring coincide with each other. Substituting Eq. (13) into Eq. (12) yields

$$(e_s)_j = -K_j\boldsymbol{b_j}\cdot\boldsymbol{a_j} + \text{const} \tag{14}$$

where the constant part of Eq. (14) is equal to $K_j(|\boldsymbol{a_j}|^2+|\boldsymbol{b_j}|^2)/2$.

Since vector $\boldsymbol{b_j}$ is fixed on link j and equals to $\boldsymbol{b_j} = (|\boldsymbol{b_j}|/|\boldsymbol{r_j}|e^{i\beta j})\boldsymbol{r_j}$, and $\boldsymbol{a_j}$ is a constant vector and equals to $\boldsymbol{a_j} = (|\boldsymbol{a_j}|/|\boldsymbol{g}|e^{iaj})\boldsymbol{g}$, Eq. (14) can be further expressed as

$$(e_s)_j = -K_j\left(\frac{|\boldsymbol{b_j}|}{|\boldsymbol{r_j}|}e^{i\beta_j}\boldsymbol{r_j}\right)\cdot\left(\frac{|\boldsymbol{a_j}|}{|\boldsymbol{g}|}e^{i\alpha_j}\boldsymbol{g}\right) + \text{const} \tag{15}$$

where $|\boldsymbol{a_j}|$ and $|\boldsymbol{b_j}|$ are the magnitudes of the position vectors of the two end points for the ground-attached spring, and α_j and β_j are the angles measured from vector $\boldsymbol{g}$ to $\boldsymbol{a_j}$ and from vector $\boldsymbol{r_j}$ to $\boldsymbol{b_j}$, respectively.

Noted that, since both $(|\boldsymbol{a_j}|/|\boldsymbol{g}|e^{i\alpha j})$ and $(|\boldsymbol{b_j}|/|\boldsymbol{r_j}|e^{i\beta j})$ are constant coefficients and based on the complex number operation, Eq. (15) can be rewritten as

$$(e_s)_j = -K_j\left(\frac{|\boldsymbol{b_j}|}{|\boldsymbol{r_j}|}e^{i\beta_j}\right)\left(\frac{|\boldsymbol{a_j}|}{|\boldsymbol{g}|}e^{-i\alpha_j}\right)\boldsymbol{g}\cdot\boldsymbol{r_j} + \text{const}$$
$$= \boldsymbol{g}\cdot\left[\left(-K_j\frac{|\boldsymbol{a_j}||\boldsymbol{b_j}|}{|\boldsymbol{r_j}||\boldsymbol{g}|}e^{i(-\alpha_j+\beta_j)}\right)\boldsymbol{r_j}\right] + \text{const} \tag{16}$$

Equation (16) indicates that the elastic potential energy $(e_s)_j$ of a ground-attached spring varies with the rotation of the link vector $\boldsymbol{r_j}$. By fitting a ground-attached spring to each ground-adjacent link, the overall potential energy of the system can be obtained, based on Eqs. (11) and (16), as

$$e_{\text{total}} = e_g + \sum_j (e_s)_j = \boldsymbol{g}\cdot\left[\sum_j\left(c''_j - K_j\frac{|\boldsymbol{a_j}||\boldsymbol{b_j}|}{|\boldsymbol{r_j}||\boldsymbol{g}|}e^{i(-\alpha_j+\beta_j)}\right)\boldsymbol{r_j}\right] + \text{const} \tag{17}$$

where j is the label of a ground-adjacent link to which a spring is attached.

In order for the overall potential energy of the linkage to be invariant at any configuration, the constant coefficient of $(c''_j - K_j(|\boldsymbol{a_j}||\boldsymbol{b_j}|/|\boldsymbol{g}||\boldsymbol{r_j}|)e^{i(-\alpha j+\beta j)})$ in Eq. (17) has to be zero, i.e.,

$$c''_j - K_j\frac{|\boldsymbol{a_j}||\boldsymbol{b_j}|}{|\boldsymbol{r_j}||\boldsymbol{g}|}e^{i(-\alpha_j+\beta_j)} = 0 \tag{18}$$

For the two complex numbers c''_j and $\boldsymbol{K_j}(|\boldsymbol{a_j}||\boldsymbol{b_j}|/|\boldsymbol{g}||\boldsymbol{r_j}|)e^{i(-\alpha j+\beta j)}$ in Eq. (18) to be equal, both their magnitudes and arguments have to be equal. Hence,

$$K_j = \frac{|c''_j||\boldsymbol{g}||\boldsymbol{r_j}|}{|\boldsymbol{a_j}||\boldsymbol{b_j}|} \tag{19}$$

and

$$\arg(c''_j) = -\alpha_j + \beta_j \tag{20}$$

Note that, referring to the ground-adjacent link in Fig. 3 and based on Eq. (20), the installing angle of the spring can be represented by $\lambda_j = \alpha_j - \beta_j - \theta_j = -\arg(c''_j) - \theta_j$. This indicates that the spring installation angle λ_j varies as the ground-adjacent link rotates and, thus, the variation in the gravitational potential energy of the system at any configuration can be fully compensated by that of the elastic potential energy of the spring if the conditions in Eqs. (19) and (20) are satisfied. Also note that, in addition to the geometry and mass properties of the linkage, the spring constant K_j in Eqs. (19) and (20) depends on the magnitudes of both spring installation vectors $\boldsymbol{a_j}$ and $\boldsymbol{b_j}$. In general, the value of spring constant K_j can be arbitrarily adjusted simply by tuning the attached ends of the spring to other corresponding position. For example, if a stiffer spring is desired, smaller magnitudes of vectors $\boldsymbol{a_j}$ and $\boldsymbol{b_j}$ have to be used. On the contrary, if a softer spring is used, larger magnitudes of vectors $\boldsymbol{a_j}$ and $\boldsymbol{b_j}$ are required. Therefore, if the gravitational potential energy of the linkage can be formulated by the expression of all ground-adjacent links, the entire gravitational potential energy of the system is fully decoupled and can be balanced by the elastic potential energy of the spring fitted on each of the ground-attached links. Since no auxiliary parallel links are used for the static balance of a linkage, such a design is considered efficient and less complex.

Table 1 Graph representations of one-DOF general SBMs up to eight links (n is the number of links)

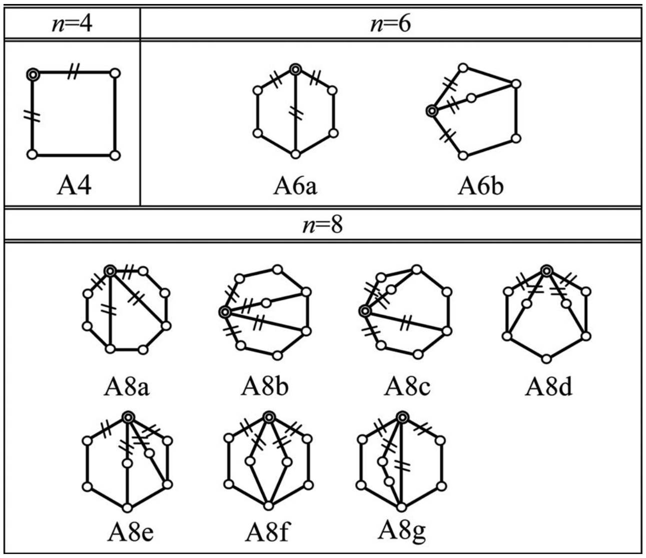
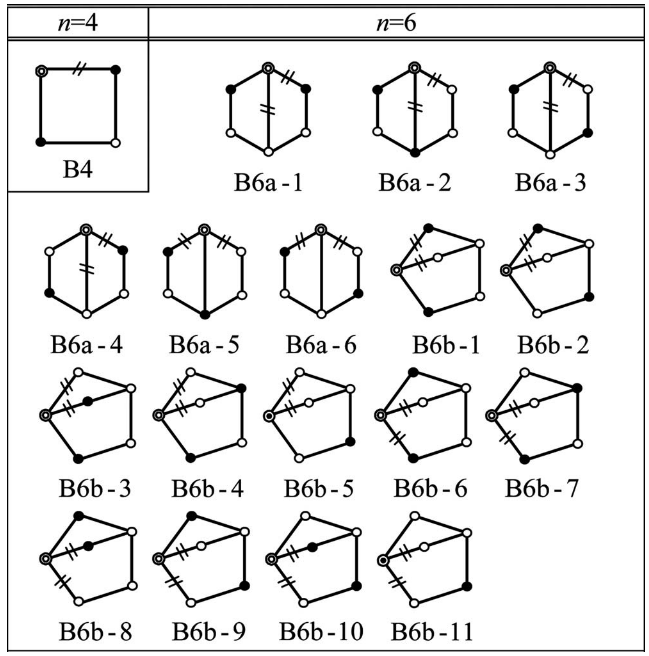


Table 2 Graph representations of one-DOF, spring balancing parallel mechanisms with ($n/2$) ground-adjacent links and ($n/2-1$) springs up to six links (n is the number of links)



5 Feasible Linkage and Its Associated Spring Installation Configuration

5.1 General Spring Balancing Linkages. A one-DOF, n-link, planar linkage with $(n/2)$ ground-adjacent links can be perfectly balanced by attaching a spring to each of the ground-adjacent links. By the use of the graph representation [22], ten feasible springs balancing four-, six-, and eight-link graphs are obtained, as illustrated in Table 1, where links are represented by vertices, joints by edges, ground link by double vertices, and springs by double slash lines. A double slash line represents that a spring is fitted between the ground vertex and its adjacent vertex. The linkages in Table 1 are referred to the SBMs of type A where the numbers of the ground-adjacent links and the variant base vectors are both equal to $(n/2)$. Note that the number and the lowercase alphabetic letters represent the number of links and distinct graph structures, respectively. For instance, graphs "A6a" and "A6b" are both six-link, type-A SBMs with different structures.

5.2 SBMs With Degenerated Number of Springs. Should less ground-attached springs are desired for the static balance of a one-DOF, n-link, planar linkage with $(n/2)$ ground-adjacent links, the number of the variant base vectors must be less than $(n/2)$. This can be achieved by imposing additional kinematic constraints into the system. Provided one parallel constraint is given as that of Eq. (6). Combining the kinematic constraint of Eq. (6) with the loop closure equations of Eq. (7) yields a total number of $(n/2)$ linearly independent equations with n vectors. The number of vectors in the basis becomes

$$\#\{B\} = n - \left(\frac{n}{2}\right) = \frac{n}{2} \tag{21}$$

Comparing Eq. (21) with Eq. (8), the number of base vectors in Eq. (21) is 1 less than that of Eq. (8), and the required number of the ground-attached springs can be reduced by 1. In the search of the admissible SBMs with parallel motion constraints, three rules are employed: First, any two adjacent links cannot be assigned as parallel motion pairs; otherwise they are considered rigidly connected and result in the degeneration of numbers of links, joints, and the degrees of freedom of the linkage; second, all kinematic isomorphic structures due to the parallel constraints are excluded; and third, since the base vectors must be linearly independent of each other, only one of the parallel motion vertices can be fitted with one ground-attached spring. As a result, the graphs of the admissible SBMs with parallel motion constraints are obtained and enumerated in Table 2, where the solid vertices represent the pair of links in parallel motion. The linkages with degenerated number of springs in Table 2 are referred to the SBMs of type B where the number of the ground-adjacent links is $(n/2)$ and the number of the variant base vectors is $(n/2-1)$. In Table 2, the graphs are labeled in the same way as those in Table 1, except for the distinction of spring configurations, e.g., "B6a-1" and "B6a-2" have the same structures, but with different spring configurations.

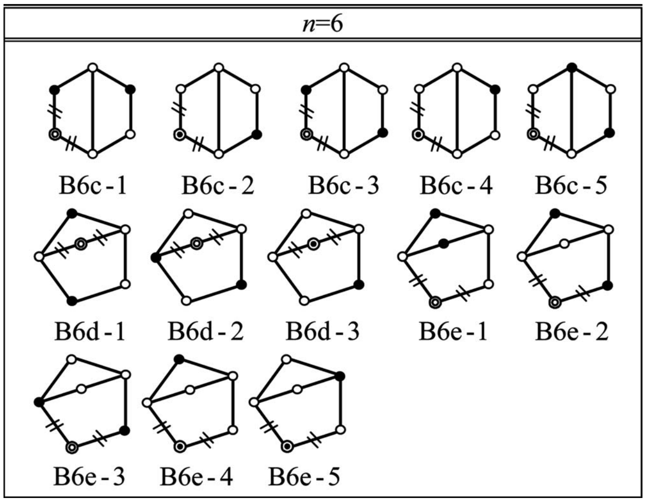
There are other SBMs of type B whose numbers of the ground-adjacent links and the variant base vectors are both $(n/2-1)$. For a one-DOF, n-link, planar linkage with $(n/2-1)$ ground-adjacent links, the admissible graphs can be obtained by kinematic inversion of the graphs in Table 2 and are illustrated in Table 3. Note that, in Table 2, the degrees of the ground vertices of the graphs are all $(n/2)$, while those in Table 3 are equal to $(n/2-1)$. Clearly, for the six-link linkages, only the Watt-I and Stephenson-I and -II mechanisms appear in Table 3. They correspond to graph numbers B6c, B6d, and B6e, respectively. Lastly, since the degrees of all vertices in a four-bar linkage equal to 2, Table 3 contains graphs of six-link linkages only.

In Tables 1–3, the SBMs are categorized into two types. Type-A linkages are referred to general SBMs containing $(n/2)$ ground-adjacent links with $(n/2)$ ground-attached springs fitted between ground and each of the ground-adjacent links. Type-B linkages are referred to SBMs with degenerated number of springs. Note that, for type-B linkage, the additional kinematic constraints have to be linearly independent of the loop closure equations.

6 Examples

6.1 Example I: Stephenson-III Type Spring Balancing Linkage. Graphs "A6a" and "A6b" in Table 1 are the Watt-II and the Stephenson-III type SBMs, respectively. Referring to the general Stephenson-III type linkage in Fig. 4, m_i represents the mass center of link i, $\boldsymbol{s}_i$ is the position vector of mass center m_i from its

Table 3 Graph representations of one-DOF, *n*-link, spring balancing parallel mechanisms with ($n/2-1$) ground-adjacent links up to six links (*n* is the number of links)



preconnected joint on link i, $\boldsymbol{a_j}$ and $\boldsymbol{b_j}$ are the position vectors of the fixed end and the link end of the spring attachment points on ground-adjacent link j, respectively, and K_j is the spring constant of the ground-attached spring on link j. Although link vector $\boldsymbol{r_i}$ can be any arbitrary vector fixed on link i, for simplicity, $\boldsymbol{r_i}$ is defined as a vector on link i between joints, as illustrated in Fig. 4. For an arbitrarily selected path from the referenced origin O, the position vector of the mass center for each link is

$$\boldsymbol{R}_2 = (p_{O,1}+1)\boldsymbol{r}_1 + p_{s,2}\boldsymbol{r}_2 \tag{22a}$$

$$\boldsymbol{R}_3 = (p_{O,1}+1)\boldsymbol{r}_1 + \boldsymbol{r}_2 + p_{s,3}\boldsymbol{r}_3 \tag{22b}$$

$$\boldsymbol{R}_4 = (p_{O,1}+1)\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 + p_{s,4}\boldsymbol{r}_4 \tag{22c}$$

$$\boldsymbol{R}_5 = (p_{O,1}+1)\boldsymbol{r}_1 + \boldsymbol{r}_2 + (1+p_{32})\boldsymbol{r}_3 + p_{s,5}\boldsymbol{r}_5 \tag{22d}$$

$$\boldsymbol{R}_6 = (p_{O,1}+1)\boldsymbol{r}_1 + \boldsymbol{r}_2 + (1+p_{32})\boldsymbol{r}_3 + \boldsymbol{r}_5 + p_{s,6}\boldsymbol{r}_6 \tag{22e}$$

where $p_{s,i}$ is the constant coefficient transforming vector $\boldsymbol{r_i}$ to vector $\boldsymbol{s_i}$, $p_{O,1}\boldsymbol{r_1}$ is the position vector from O to the fixed pivot of link 4, and complex numbers p_{32} and p_{12} are the constant transformation coefficients for the ternary links 1 and 3, respectively.

According to Eqs. (3) and (4), the overall gravitational potential energy is the sum of the gravitational potential energy of each link. By substituting Eq. (22) into Eq. (3), the expressions for the gravitational potential energy of the system are expressed in terms of six-link vectors as Eq. (4) as

$$e_g = \boldsymbol{g}\cdot\left(\sum_{i=1}^{6} c_i\boldsymbol{r_i}\right) \tag{23}$$

where

$$c_1 = (1+p_{O,1})(-m_2-m_3-m_4-m_5-m_6) \tag{24a}$$

$$c_2 = -m_2p_{s,2} - m_3 - m_4 - m_5 - m_6 \tag{24b}$$

$$c_3 = -m_4 + (1+p_{32})(-m_5-m_6) - m_3p_{s,3} \tag{24c}$$

$$c_4 = -m_4p_{s,4} \tag{24d}$$

$$c_5 = -m_6 - m_5p_{s,5} \tag{24e}$$

$$c_6 = -m_6p_{s,6} \tag{24f}$$

The two linearly independent loop closure equations of the Stephenson-III type linkage are given as

$$\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 + \boldsymbol{r}_4 = 0 \tag{25a}$$

and

$$p_{12}\boldsymbol{r}_1 - \boldsymbol{r}_4 + p_{32}\boldsymbol{r}_3 + \boldsymbol{r}_5 + \boldsymbol{r}_6 = 0 \tag{25b}$$

Choosing the vectors of ground and the ground-adjacent links as the base vectors, vectors $\boldsymbol{r}_3$ and $\boldsymbol{r}_5$ can be expressed in terms of the base vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_4$, and $\boldsymbol{r}_6$. By collecting the variant base vectors terms, the gravitational potential energy in Eq. (23) can be rewritten as

$$e_g = \boldsymbol{g}\cdot\left(\sum_{j=2,4,6} c_j''\boldsymbol{r_j}\right) + \text{const} \tag{26}$$

where

$$c_2'' = -m_3 + m_5p_{32} - m_2p_{s,2} + m_3p_{s,3} - m_5p_{32}p_{s,5} \tag{27a}$$

$$c_4'' = m_4 + m_5 + m_5p_{32} + m_3p_{s,3} - m_4p_{s,4} - m_5p_{s,5} - m_5p_{32}p_{s,5} \tag{27b}$$

and

$$c_6'' = m_5p_{s,5} + m_6 - m_6p_{s,6} \tag{27c}$$

Note that, since $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_4$, and $\boldsymbol{r}_6$ are chosen as the base vectors, the constant coefficients c_2'', c_4'', and c_6'' are uniquely determined. Since the term $\boldsymbol{g}\cdot(c_1''\boldsymbol{r}_1)$ in Eq. (26) is constant and irrelevant to the spring design conditions, it is not expressed explicitly. By fitting a ground-attached springs between ground and links 2, 4, and 6, respectively, as shown in Fig. 4, the balance of the total potential energy of the system can be accomplished, and the spring constants and the spring attachment points for each of the ground-attached springs can be determined based on Eqs. (19) and (20) as

$$K_j = \frac{|c_j''||\boldsymbol{g}||\boldsymbol{r_j}|}{|\boldsymbol{a_j}||\boldsymbol{b_j}|}, \quad j=2,4,6 \tag{28a}$$

and

$$\arg(c_j'') = -\alpha_j + \beta_j, \quad j=2,4,6 \tag{28b}$$

The spring constants K_2, K_4, and K_6 and the spring design parameters $|\boldsymbol{a}_2|$, $|\boldsymbol{a}_4|$, $|\boldsymbol{a}_6|$, $|\boldsymbol{b}_2|$, $|\boldsymbol{b}_4|$, $|\boldsymbol{b}_6|$, α_2, α_4, α_6, β_2, β_2, and β_2 can be selected according to Eq. (28). Consider a linkage with given geometry and mass properties provided in Table 4 where the initial configuration of the linkage represented by vectors $\boldsymbol{r_j}$ and $\boldsymbol{s_j}$ for $j=1,2,\ldots,6$ is also included. With predetermined values of spring constants $K_2=1000$ N/m, $K_4=3000$ N/m, and $K_6=5000$ N/m, the spring attachment points of the three springs are readily obtained via Eqs. (19) and (20) and listed in Table 5.

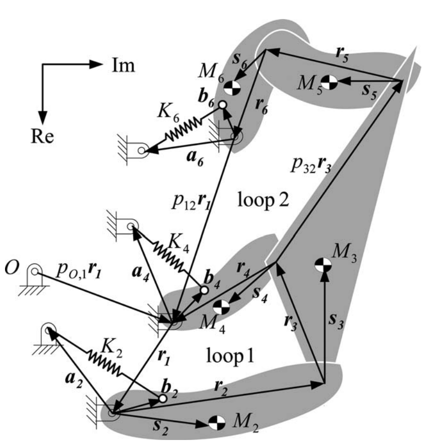


Fig. 4 Stephenson-III type linkage with three ground-attached springs

Table 4 Geometry and mass properties of the Stephenson-III type linkage

Link i	Geometry properties (m)		Mass properties (Kg, m)	
1	$\boldsymbol{r}_1=0.9i$;	$p_{12}=0.114-0.125i$;		
2	$\boldsymbol{r}_2=-0.416+0.422i$;		$m_2=1.396$;	$\boldsymbol{s}_2=-0.208+0.211i$;
3	$\boldsymbol{r}_3=0.334-0.893i$;	$p_{32}=0.108-0.187i$;	$m_3=5.065$;	$\boldsymbol{s}_3=0.167-0.608i$;
4	$\boldsymbol{r}_4=0.082-0.450i$;		$m_4=0.821$;	$\boldsymbol{s}_4=0.041-0.225i$;
5	$\boldsymbol{r}_5=0.200+0.346i$;		$m_5=20.946$;	$\boldsymbol{s}_5=-0.046+1.398i$;
6	$\boldsymbol{r}_6=-0.100-0.534i$;		$m_6=1.738$;	$\boldsymbol{s}_6=-0.050-0.267i$;

Similarly, the design of a spring balancing Watt-II linkage can be implemented using this methodology applicable to the design of the Stephenson-III type linkage.

6.2 Example II: Spring Balancing Parallelogram Four-Bar Linkage. The four-bar parallelogram of graph B4 in Table 2 is the simplest structure of a parallel linkage. As shown in Fig. 5, due to its special geometry, links 2 and 4 are always parallel to each other, and the kinematic relation between links 2 and 4 is

$$\boldsymbol{r}_4 = -\boldsymbol{r}_2 \tag{29}$$

The coupler link 3 is in circular translation and its orientation is parallel with respect to ground. According to Eq. (11), the gravitational potential energy of the parallelogram four-bar linkage is

$$e_g = \boldsymbol{g} \cdot \left(\sum_{j=2,4} c''_j \boldsymbol{r_j} \right) + \text{const} \tag{30}$$

where

$$c''_2 = -m_2 p_{s,2} - m_3 + m_3 p_{s,3} \tag{31a}$$

and

$$c''_4 = -m_4 p_{s,4} + m_4 + m_3 p_{s,3} \tag{31b}$$

Equation (29) is considered as the additional kinematic constraint to a general four-bar linkage. Substituting Eq. (29) into Eq. (30) yields

$$e_g = \boldsymbol{g} \cdot (c'''_2 \boldsymbol{r}_2) + \text{const} \tag{32}$$

where

$$c'''_2 = -m_2 p_{s,2} - m_3 - m_4 + m_4 p_{s,4} \tag{33}$$

Table 5 Spring design parameters

Spring on link j	Stiffness (N/m)	Attachment points (m)	
2	1000	$\boldsymbol{a}_2=-0.173-0.040i$;	$\boldsymbol{b}_2=-0.334+0.337i$;
4	3000	$\boldsymbol{a}_4=-0.202+0.144i$;	$\boldsymbol{b}_4=0.068-0.340i$;
6	5000	$\boldsymbol{a}_6=0.155-0.094i$;	$\boldsymbol{b}_6=-0.078-0.428i$;

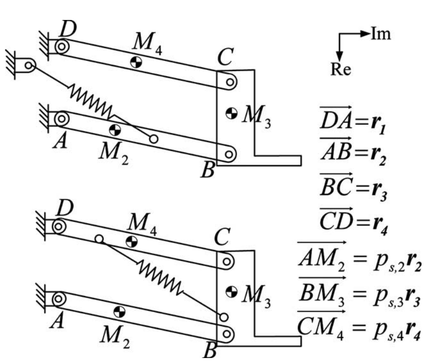


Fig. 5 Spring balancing parallelogram four-bar linkages

Equation (32) shows that the gravitational potential energy of the parallelogram four-bar linkage can be described by one variant base vector $\boldsymbol{r}_2$; thus, the system requires only one ground-attached spring for the static balance of the system. The spring design conditions are obtained by replacing the constant coefficient c''_j with c'''_2 in Eqs. (19) and (20). In comparison with a general four-bar linkage, the number of springs required in a statically balanced parallelogram four-bar linkage is degenerated by 1. It is to be noted that, for a parallelogram four-bar linkage, the spring can be fitted between ground and link 2 or between links 4 (or 2) and 3. Because of the parallel motion, the coupler link, link 3, provides a "pseudo" ground at the distal end. This is the main reason for many designers to adopt parallel auxiliary links in the spring balancing linkages.

6.3 Example III: Spring Balancing Watt-I Parallel Motion Generator. Another example of a one-DOF linkage with parallel motion is a Watt-I parallel motion generator [23]. Referring to graph B6c-4 in Table 3, the dimensions of the six-link, Watt-I linkage are constructed, as shown in Fig. 6, where link 5 and ground link are the parallel pairs. Link 5 always moves in parallel with respect to ground and undergoes curvilinear translation. Again, as depicted in Eq. (4), the gravitational potential energy of the Watt-I six-link linkage can be expressed in terms of six link vectors, $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_6$, and the loop closure equations of the linkage are

$$\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 + \boldsymbol{r}_4 = 0 \tag{34a}$$

and

$$p_{42}\boldsymbol{r}_4 + p_{32}\boldsymbol{r}_3 + \boldsymbol{r}_5 + \boldsymbol{r}_6 = 0 \tag{34b}$$

From Eq. (8), the parallelism between link 5 and ground is expressed as

$$\boldsymbol{r}_5 = q\boldsymbol{r}_1 \tag{35}$$

where

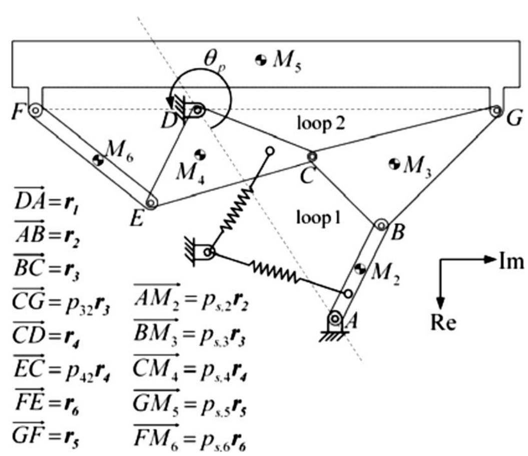


Fig. 6 Watt-I parallel motion generator with two springs

$$q = \frac{|\boldsymbol{r}_5|}{|\boldsymbol{r}_1|} e^{i\theta_p} \tag{36}$$

and θ_p is the relative orientation angle of link 5 with respect to ground.

Since the kinematic constraints of the loop closure equations of Eq. (34) and the parallel motion of Eq. (35) are linearly independent in general, the two vectors, $\boldsymbol{r}_3$ and $\boldsymbol{r}_6$, can be obtained as linear combination of the base vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, and $\boldsymbol{r}_4$ as

$$\boldsymbol{r}_3 = -\boldsymbol{r}_1 - \boldsymbol{r}_2 - \boldsymbol{r}_4 \tag{37a}$$

and

$$\boldsymbol{r}_6 = (-q + p_{32})\boldsymbol{r}_1 + p_{32}\boldsymbol{r}_2 + (p_{32} - p_{42})\boldsymbol{r}_4 \tag{37b}$$

Substituting Eqs. (35) and (37) into Eq. (4), the gravitational potential energy of the six-link Watt-I linkage is obtained as

$$e_g = \boldsymbol{g} \cdot \left(\sum_{j=2,4} c_j''' \boldsymbol{r}_j \right) + \text{const} \tag{38}$$

where

$$c_2''' = -m_3 - m_2 p_{s,2} + m_3 p_{s,3} + m_5 p_{32} + m_6 p_{32} - m_6 p_{32} p_{s,6} \tag{39a}$$

and

$$c_4''' = m_4 + m_5 + m_6 + m_3 p_{s,3} + m_5 p_{32} + m_6 p_{32} - m_4 p_{s,4} - m_6 p_{32} p_{s,6} + m_6 p_{42} p_{s,6} \tag{39b}$$

Fitting a ground-attached spring to each of ground-adjacent links 2 and 4, respectively, and followed by the cancellation of the variations in the gravitational potential energy and the elastic potential energy of the system, the design parameters of the spring can be obtained via Eq. (18). This example demonstrates that the design of a statically balanced one-DOF, n-link, planar parallel linkage with $(n/2-1)$ ground-adjacent link can be accomplished by using degenerated number of springs.

The motions of the Stephenson-III type six-bar linkage of Fig. 4, the parallelogram four-bar linkage of Fig. 5, and the Watt-I type six-bar linkage of Fig. 6 are simulated by ADAMS, and the gravitational, elastic, and total potential energies versus crank angle for an entire cycle are plotted in Figs. 7(*a*)–7(*c*), respectively. Figure 7 shows that the variations in the gravitational potential energies of the three systems can be perfectly compensated by that of the elastic potential energies of the ground-attached springs.

In general, a Watt-I linkage with only two ground-adjacent links cannot be statically balanced with ground-attached springs alone. However, due to the inclusion of the kinematic constraint of parallel motion, the number of the base vectors required in the formulation of the gravitational potential energy of the Watt-I linkage is degenerated by 1. Hence, only two ground-attached springs are needed. The concept of degeneration of the linearly independent vectors in the formulation of the gravitational potential energy of the planar linkage can be extended to linkages of higher links. With the aid of graph representations, more feasible higher-link SBMs can be enumerated systematically. In practice, static balance of a one-DOF, n-link planar linkage with less than $(n/2)$ ground-adjacent links can be obtained, as long as the kinematic constraint equations are linearly independent of the existed loop closure equations.

7 Conclusion

A systematic methodology for the design of a full-cycle, statically balanced, one-DOF planar linkage with revolute joints only is presented. In the methodology, no auxiliary parallel links are required. Design of a statically balanced, one-DOF, n-link, planar linkage with $(n/2)$ ground-adjacent links can be accomplished by fitting a ground-attached spring to each of the ground-adjacent links. Admissible graphs of one-DOF planar linkages of up to eight links are enumerated. It is also shown that the SBMs with degenerated number of the springs can be obtained, provided that the prescribed kinematic constraints of the linkage are linearly independent of the inherited loop closure equations. This methodology is successfully demonstrated by the designs of the static balances of a general Stephenson-III type six-bar linkage with $(n/2)$ ground-adjacent links and a Watt-I type six-bar linkage of parallel motion with $(n/2-1)$ ground-adjacent links. Results are simulated and justified by the software ADAMS.

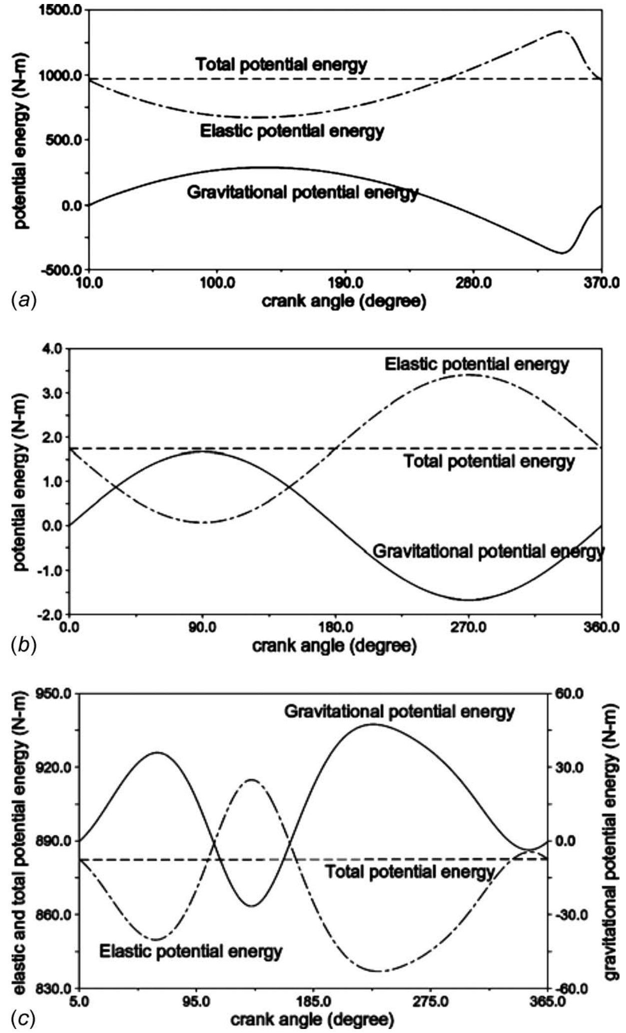


Fig. 7 Potential energy curves of (*a*) the Stephenson-III linkage, (*b*) the parallelogram linkage, and (*c*) the Watt-I parallel motion generator

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