



A stiffness matrix approach for the design of statically balanced planar articulated manipulators

Po-Yang Lin^a, Win-Bin Shieh^b, Dar-Zen Chen^{a,*}

^a Dept. of Mechanical Engineering, National Taiwan University, Taipei, Taiwan

^b Dept. of Mechanical Engineering, Mingchi University of Technology, Taipei, Taiwan

ARTICLE INFO

Article history:

Received 24 February 2010

Received in revised form 4 August 2010

Accepted 7 August 2010

Keywords:

Static balance

Zero-free-length spring

Articulated manipulator

Stiffness block matrix

ABSTRACT

A methodology is developed to determine the spring installation for the design of a statically balanced planar articulated manipulator without parallel auxiliary links. The spring installation is characterized by the connectivity of springs among links, the selection of spring constants, and the locations of spring attachment points. The static equilibrium analysis of the spring-loaded planar articulated manipulator is based on the energy approach, formulated by a constant stiffness block matrix and its associated configuration block matrices. The stiffness block matrix quantifies the resistance or assistance of a manipulator to the change of configuration due to the gravitational forces and the elastic spring forces. Such a matrix uniquely represents both the gravitational potential energy and the elastic potential energy of springs of the system at any configuration. By solving the isotropic condition of the stiffness block matrix, all design parameters of springs can be obtained for any given planar articulated manipulator with prescribed dimensions and inertia. Exact solutions for the locations of attachment points are given in detailed in the examples of a spring-loaded one-, two- and three- degrees of freedom articulated manipulators.

© 2010 Elsevier Ltd. All rights reserved.

1. Introduction

A mechanism is said to be perfectly gravity-balanced if the weight of the links of the mechanism does not produce any torque at the kinematic pair under static conditions, for any configuration of the mechanism within its workspace [1]. Gravity-balanced planar manipulators can benefit from the increase of the power efficiency since no actuating force is required to sustain the system payload. These manipulators have a wide range of applications on support apparatuses [2–15]. For example, a surgical tool can be posed statically at any spontaneous posture by a gravity-balanced manipulator arm [8,16].

In the field of statically balanced mechanisms, a large number of designs of planar articulated manipulators are embedded with parallel auxiliary links and elastic elements, e.g. springs [8–15,17–22]. Such springs produce an equal but opposite torque at each revolute joint to that produced by the weight of manipulator links at any configuration to achieve the perfect balance. The parallel auxiliary links provide the suitable locations of spring attachment points. The design approach is easy to apply for planar articulated manipulators. However, the parallelogram structures of such designs have several disadvantages: (i) motion interferences between the primary links and the auxiliary links may hinder the workspace of the manipulator; (ii) mechanical tolerances must be carefully controlled to maintain the parallelism of auxiliary links; (iii) auxiliary links produce additional inertia to the system; and (iv) the excess number of parts may reduce the robustness of the manipulator. More importantly, besides the auxiliary link method, a designer has almost no foundation to determine the spring installation for a general planar articulated manipulator.

The nature of static balance of a system with springs is the energy exchanged equally between the gravitational potential energy and the elastic potential energy. The gravitational forces, like spring forces, resist or assist the configuration change of a

* Corresponding author. Fax: +886 2 23631755.

E-mail address: dzchen@ntu.edu.tw (D.-Z. Chen).

mechanism from an equilibrium configuration. In this paper, a generalized stiffness of mechanism based on the derivation of potential energy is proposed. Such a generalized stiffness is formulated in a constant block matrix, referred to as a *stiffness block matrix*, and is used for the analysis of static equilibrium. Traditionally, the stiffness matrix of a manipulator is derived from the coordinate transformations between the actuator space and the end effector space [23,24]. Such stiffness matrix is a relation between the end effector output forces and joint actuating torques, and it is configuration dependent [23,24]. The stiffness block matrix of this paper is, however, configuration independent and constant for a mechanism with given dimensions and inertia. Such quantity is physically close to the sense of the spring constant of Hooke's law.

In this paper, a methodology is developed to determine the spring installation for the design of statically balanced planar articulated linkage without parallel auxiliary links. The spring installation is characterized by the connectivity of springs among links, the selection of spring constants, and the locations of spring attachment points. The connectivity of springs among links is referred to as the spring configuration. The spring configuration can be determined by the structure of a stiffness block matrix. The locations of spring attachment points can be derived from the design equations directly obtained from the component matrices in the stiffness block matrix. Since the configuration of mechanism is defined by each orientation of link, spring deformation is associated with the rotational motions of links; manipulators containing prismatic joints are excluded from this study. In the final part of this paper, exact solutions for the locations of attachment points are given with details in the examples of a spring-loaded one-, two- and three- degree of freedom (DOF) articulated manipulators.

2. Stiffness block matrix representation of elastic potential energy

In a mechanical system, a stiffness matrix $\mathbf{K} = [k_{ij}]$ is used to express the potential energy U of the mechanical system displacing from an equilibrium position, by means of the following equation:

$$U = \frac{1}{2} \mathbf{Q}^T \mathbf{K} \mathbf{Q} \quad (1)$$

where most often, \mathbf{Q} is the matrix representation of a vector whose components are the generalized coordinates of the system, e.g. $\mathbf{Q} = [\theta_1, \theta_2, \dots, \theta_n]$ for θ_i as the joint angle of joint i .

Eq. (1) is widely used to analyze the equilibriums for some mechanical systems with joint compliances behaving like torsion springs. However, mechanisms with linear spring installation are sometimes difficult to apply by the inability to preserve the bilinearity of the formulation. Hence, if quantity θ_i describes the angular displacement of a moving link with respect to a fixed link in a planar linkage, a unit vector $\boldsymbol{\theta}_i$ can be defined as

$$\boldsymbol{\theta}_i = (C\theta_i)\mathbf{e}_1 + (S\theta_i)\mathbf{e}_2 \quad (2)$$

where abbreviations $C\theta$ and $S\theta$ are denoted for $\cos(\theta)$ and $\sin(\theta)$, respectively, throughout the paper for the conciseness, and \mathbf{e}_1 and \mathbf{e}_2 are the orthonormal vectors spanning a Cartesian plane XY , namely the working plane of the planar linkage.

For a rigid body i , vector $\boldsymbol{\theta}_i$ is an arbitrary unit vector fixed on the rigid body. Angle θ_i can be specifically measured counterclockwise from the positive X -axis of an inertia frame to the positive direction of the unit vector. The matrix representation of vector $\boldsymbol{\theta}_i$ is

$$\mathbf{Q}_i = [C\theta_i \quad S\theta_i]^T \quad (3)$$

By Eq. (3), the orientation of the rigid body i on plane XY can be expressed uniquely. For a vector \mathbf{r} of length r fixed to the same rigid body i , the matrix representation \mathbf{R} for vector \mathbf{r} can be obtained by pre-multiplying \mathbf{Q}_i with a constant transformation matrix \mathbf{T} :

$$\mathbf{T}(r, \varphi) = \begin{bmatrix} rC\varphi & -rS\varphi \\ rS\varphi & rC\varphi \end{bmatrix} \quad (4)$$

where φ is the relative angle, measured counterclockwise, between vector \mathbf{r} and the unit vector $\boldsymbol{\theta}_i$, and angle φ is constant at any configuration of the rigid body i .

Consider a zero-free-length spring with labeled number z , spring constant k_z and its two ends attached arbitrarily to links x and y of a serial kinematic chain as illustrated in Fig. 1, where link x is closer to the ground than link y . An ideal zero-free-length spring can be made up with a pre-tensioned, normal spring and possibly with cables and pulleys [19–21]. The vectors of interest in Fig. 1 are \mathbf{a}_z , \mathbf{b}_z and \mathbf{r}_i , and they can be respectively represented by matrices \mathbf{A}_z , \mathbf{B}_z and \mathbf{R}_i as

$$\mathbf{A}_z = \mathbf{T}(a_z, \alpha_z)\mathbf{Q}_x \quad (5a)$$

$$\mathbf{B}_z = \mathbf{T}(b_z, \beta_z)\mathbf{Q}_y \quad (5b)$$

$$\mathbf{R}_i = \mathbf{T}(r_i, \varphi_i)\mathbf{Q}_i \quad (5c)$$

where matrices \mathbf{A}_z and \mathbf{B}_z contain the positioning parameters of the spring attachment points on links x and y , respectively, and vector \mathbf{r}_i is fixed on link i and pointing from one center of revolute joint to the other.

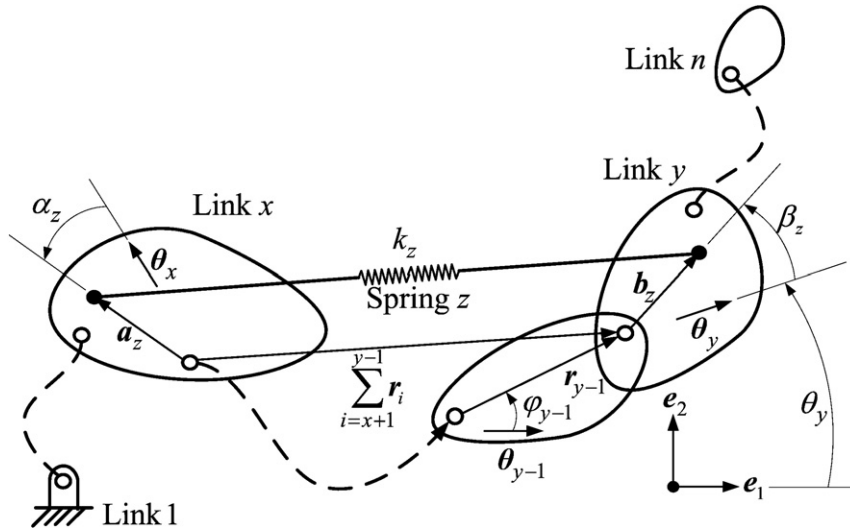


Fig. 1. A spring z attached to links x and y.

For consistency throughout this paper, capital boldface, lowercase boldface and lowercase italic characters are used for matrices, vectors and scalars, respectively.

The elastic potential energy zU of spring z can be obtained by the spring constant times the square of the spring deformation as

$${}^zU = \frac{1}{2}k_z \left| \sum_{i=x+1}^{y-1} \mathbf{r}_i + \mathbf{b}_z - \mathbf{a}_z \right|^2 = \frac{1}{2}k_z \left(\sum_{i=x+1}^{y-1} \mathbf{R}_i^T + \mathbf{B}_z^T - \mathbf{A}_z^T \right) \left(\sum_{i=x+1}^{y-1} \mathbf{R}_i + \mathbf{B}_z - \mathbf{A}_z \right) \quad (6)$$

By substituting Eqs. (5a)–(5c) into Eq. (6), a bilinear form can be obtained as

$${}^zU = \frac{1}{2} \sum_{i,j=x}^y \mathbf{Q}_i^T ({}^z\mathbf{E}_{ij}) \mathbf{Q}_j \quad (7)$$

where matrix ${}^z\mathbf{E}_{ij}$ with indices $i \neq j$ is

$${}^z\mathbf{E}_{ij} = \begin{cases} -k_z \mathbf{A}_z^T \mathbf{B}_z = -k_z \mathbf{T}^T(a_z, \alpha_z) \mathbf{T}(b_z, \beta_z) & i = x; j = y \\ -k_z \mathbf{A}_z^T \mathbf{R}_i = -k_z \mathbf{T}^T(a_z, \alpha_z) \mathbf{T}(r_i, \varphi_i) & i = x; j = (x + 1), \dots, (y - 1) \\ k_z \mathbf{R}_i^T \mathbf{B}_z = k_z \mathbf{T}^T(r_i, \varphi_i) \mathbf{T}(b_z, \beta_z) & i = (x + 1), \dots, (y - 1); j = y \\ k_z \mathbf{R}_i^T \mathbf{R}_j = k_z \mathbf{T}^T(r_i, \varphi_i) \mathbf{T}(r_j, \varphi_j) & i, j \neq x, y; j > i \end{cases} \quad (8)$$

and

$${}^z\mathbf{E}_{ji} = {}^z\mathbf{E}_{ij}^T \quad (9)$$

For matrix ${}^z\mathbf{E}_{ij}$ with indices $i = j$,

$${}^z\mathbf{E}_{ii} = \begin{cases} k_z \mathbf{A}_z^T \mathbf{A}_z = k_z a_z^2 \mathbf{I}, & i = x \\ k_z \mathbf{R}_i^T \mathbf{R}_i = k_z r_i^2 \mathbf{I}, & i = x + 1, \dots, y - 1 \\ k_z \mathbf{B}_z^T \mathbf{B}_z = k_z b_z^2 \mathbf{I}, & i = y \end{cases} \quad (10)$$

where \mathbf{I} is the 2×2 identity matrix.

Hence, a block matrix form of Eq. (7) can be expressed as

$${}^zU = \frac{1}{2} \begin{bmatrix} \mathbf{Q}_x \\ \mathbf{Q}_{x+1} \\ \vdots \\ \mathbf{Q}_{y-1} \\ \mathbf{Q}_y \end{bmatrix}^T \begin{bmatrix} {}^z\mathbf{E}_{xx} & {}^z\mathbf{E}_{x(x+1)} & \dots & {}^z\mathbf{E}_{x(y-1)} & {}^z\mathbf{E}_{xy} \\ {}^z\mathbf{E}_{x(x+1)}^T & {}^z\mathbf{E}_{(x+1)(x+1)} & & & {}^z\mathbf{E}_{(x+1)y} \\ \vdots & & \ddots & & \vdots \\ {}^z\mathbf{E}_{x(y-1)}^T & & & {}^z\mathbf{E}_{(y-1)(y-1)} & {}^z\mathbf{E}_{(y-1)y} \\ {}^z\mathbf{E}_{xy}^T & {}^z\mathbf{E}_{(x+1)y}^T & \dots & {}^z\mathbf{E}_{(y-1)y}^T & {}^z\mathbf{E}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_x \\ \mathbf{Q}_{x+1} \\ \vdots \\ \mathbf{Q}_{y-1} \\ \mathbf{Q}_y \end{bmatrix} \quad (11)$$

Eq. (11) can be rewritten in a block matrix form as

$${}^zU = \frac{1}{2} [\mathbf{Q}_i]^T [{}^z\mathbf{K}_{ij}] [\mathbf{Q}_i] = \frac{1}{2} \mathcal{Q}^T ({}^z\mathcal{K}) \mathcal{Q} \quad (12)$$

where $[{}^z\mathbf{K}_{ij}]$ is a $2n \times 2n$ block matrix and is denoted as ${}^z\mathcal{K}$, $[\mathbf{Q}_i]$ is a $2n \times 1$ block matrix and is denoted as $\mathcal{Q} = [\mathbf{Q}_1, \mathbf{Q}_2, \dots, \mathbf{Q}_n]^T$, while ${}^z\mathbf{K}_{ij}$ is a 2×2 component matrix, for $i, j = 1, 2, \dots, n$, as

$${}^z\mathbf{K}_{ij} = \begin{cases} {}^z\mathbf{E}_{ij} & \text{for } i, j = x, (x+1), \dots, y \\ \mathbf{0} & \text{else} \end{cases} \quad (13)$$

Block matrix ${}^z\mathcal{K}$ is referred to as the stiffness block matrix of spring z . The stiffness block matrix has $n \times n$ stiffness components ${}^z\mathbf{K}_{ij}$'s, and $2n \times 2n$ entries. For spring z is attached to links x and y , the stiffness block matrix has $(y-x+1) \times (y-x+1)$ non-zero elastic components \mathbf{E}_{ij} 's. Denote the installation configuration of the spring z as $z(x, y)$. The resultant stiffness block matrix for spring $z(x, y)$ is constant and uniquely determined by parameters k_z, a_z, α_z, b_z and β_z . The associated block matrix \mathbf{Q} is referred to as the configuration block matrix of a planar manipulator. The configuration block matrix has $n \times 1$ matrix components \mathbf{Q}_i 's, and $2n \times 1$ scalar components. The configuration block matrix is a geometrical measure of configuration change by the relative angular displacement between any two links, including the moving links and the fixed link, in a planar linkage. Each configuration has a unique configuration block matrix by the given angular displacement θ_i of each articulated joint i . Elastic potential energy by spring z for each configuration can be determined according to Eq. (12).

3. Stiffness block matrix representation of gravitational potential energy

Gravitational potential energy of a mechanism varies along with the configuration change of the mechanism. Consequently, the gravitational force behaves physically as the spring force resisting or assisting the configuration change of the mechanism from an equilibrium configuration. Hence, an intuitive idea is motivated to formulate the gravitational potential energy in the matched form as that of Eq. (12). Due to the same reason that the bilinear form by the conventional basis $\mathbf{Q} = [\theta_1, \theta_2, \dots, \theta_n]$ is difficult to be obtained, configuration block matrix \mathcal{Q} is employed.

As illustrated in Fig. 2, the mass center position vector \mathbf{p}_x of link x in a planar linkage can be obtained along a path passing all centers of pre-connected revolute joints. The path starts from an origin of a fixed frame on the ground link, passes through each link i , and then ends on link x . Thus, the mass center position vector \mathbf{p}_x can be represented by matrix \mathbf{P}_x as

$$\mathbf{P}_x = \sum_{i=1}^{x-1} \mathbf{T}(r_i, \varphi_i) \mathbf{Q}_i + \mathbf{T}(s_x, \sigma_x) \mathbf{Q}_x \quad (14)$$

where $\mathbf{T}(s_x, \sigma_x) \mathbf{Q}_x$ represents the vector pointing from the center of joint connecting links $(x-1)$ and x to the mass center of link x , as shown in Fig. 2.

The gravitational potential energy of link x can be obtained by

$${}^gU_x = -m_x \mathbf{G}^T \mathbf{P}_x \quad (15)$$

where \mathbf{G} is a 2×1 column matrix representing the gravitational acceleration vector \mathbf{g} of magnitude g , and is fixed to ground (link 1), hence,

$$\mathbf{G} = [gC(\theta_1 + \gamma) \quad gS(\theta_1 + \gamma)]^T = \mathbf{T}(\mathbf{g}, \gamma) \mathbf{Q}_1 \quad (16)$$

The total gravitational potential energy is the sum of Eq. (15) on index x as

$${}^gU = -\mathbf{G}^T \sum_x (m_x \mathbf{P}_x) \quad (17)$$

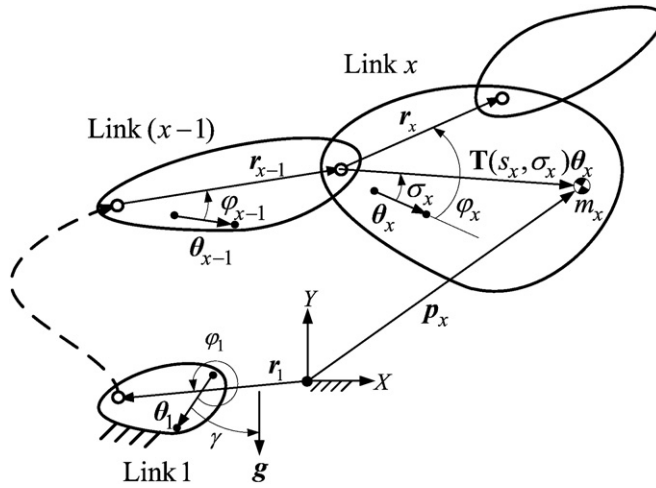


Fig. 2. The mass center position vector of link x .

According to Eq. (14), the sum of $m_x \mathbf{R}_x$ for an n -link manipulator, $x = 1, 2, \dots, n$, can be obtained as

$$\sum_{i=1}^n m_i \mathbf{P}_i = \sum_{i=1}^n \left[m_i \mathbf{T}(s_i, \sigma_i) + \left(\sum_{j=i+1}^n m_j \right) \mathbf{T}(r_i, \varphi_i) \right] \mathbf{Q}_i \tag{18}$$

Substituting Eqs. (16) and (18) into Eq. (17) yields

$${}^g U = \sum_{i=1}^n \mathbf{Q}_i^T \mathbf{D}_i \mathbf{Q}_i \tag{19}$$

where

$$\mathbf{D}_i = -\mathbf{T}^T(\mathbf{g}, \gamma) \left[m_i \mathbf{T}(s_i, \sigma_i) + \left(\sum_{j=i+1}^n m_j \right) \mathbf{T}(r_i, \varphi_i) \right] \tag{20}$$

Matrix \mathbf{D}_i is a 2×2 matrix

$$\mathbf{D}_i = \begin{bmatrix} -d_{i,1} & d_{i,2} \\ -d_{i,2} & -d_{i,1} \end{bmatrix} \tag{21}$$

where

$$d_{i,1} = m_i g s_i C(\sigma_i - \gamma) + \left(\sum_{j=i+1}^n m_j \right) g r_i C(\varphi_i - \gamma) \tag{22a}$$

$$d_{i,2} = m_i g s_i S(\sigma_i - \gamma) + \left(\sum_{j=i+1}^n m_j \right) g r_i S(\varphi_i - \gamma) \tag{22b}$$

Since $(\mathbf{Q}_i^T \mathbf{D}_i \mathbf{Q}_i)$ is a scalar, it equals to its transpose $(\mathbf{Q}_i^T \mathbf{D}_i^T \mathbf{Q}_i)$. The bilinear form of Eq. (19) can be expressed to a block matrix form as

$${}^g U = \frac{1}{2} \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \vdots \\ \mathbf{Q}_n \end{bmatrix}^T \begin{bmatrix} 2\mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 & \dots & \mathbf{D}_n \\ \mathbf{D}_2^T & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \mathbf{D}_3^T & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{D}_n^T & \mathbf{0} & \dots & \dots & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \vdots \\ \mathbf{Q}_n \end{bmatrix} \tag{23}$$

Hence, the stiffness block matrix ${}^g \mathcal{K}$ by the gravitational effects can be obtained as

$${}^g U = \frac{1}{2} [\mathbf{Q}_i]^T [{}^g \mathbf{K}_{ij}] [\mathbf{Q}_i] = \frac{1}{2} \mathcal{Q}^T ({}^g \mathcal{K}) \mathcal{Q} \tag{24}$$

where each component matrix ${}^g\mathbf{K}_{ij}$ of the $2n \times 2n$ ${}^g\mathcal{K}$ block matrix is

$${}^g\mathbf{K}_{ij} = \begin{cases} 2\mathbf{D}_1 & i = 1, j = 1 \\ \mathbf{D}_j & i = 1, j = 2, 3, \dots, n \\ \mathbf{D}_i^T & j = 1, i = 2, 3, \dots, n \\ \mathbf{0} & \text{else} \end{cases} \quad (25)$$

Matrix \mathbf{D}_i is referred to as a gravitational component in the stiffness block matrix. According to Eqs. (21), (22a) and (22b), a gravitational component \mathbf{D}_i contains the dimensional and inertia parameters of the given planar manipulator. These parameters affect the gravitational forces or moments acting on the planar manipulator. A given manipulator has a constant and unique stiffness block matrix ${}^g\mathbf{K}$ that characterize the effects of gravity at different configurations.

4. Stiffness block matrix under force isotropy condition

Assume that several springs are attached to a planar n -link manipulator to achieve a perfect gravity balance. The total potential energy can be expressed by the use of Eqs. (12) and (24) as

$$U = {}^gU + \sum_z {}^zU = \frac{1}{2} \mathcal{Q}^T \mathcal{K} \mathcal{Q} \quad (26)$$

where the overall stiffness block matrix \mathcal{K} is

$$\mathcal{K} = [\mathbf{K}_{ij}] = [{}^g\mathbf{K}_{ij}] + \sum_z [{}^z\mathbf{K}_{ij}] \quad (27)$$

and $\mathbf{K}_{ji} = \mathbf{K}_{ij}^T$.

Substituting Eq. (27) with Eqs. (13) and (25) yields

$$\mathbf{K}_{ij} = \begin{cases} 2\mathbf{D}_1 + \sum_z {}^z\mathbf{E}_{11} & i = 1, j = 1 \\ \mathbf{D}_j + \sum_z {}^z\mathbf{E}_{ij} & i = 1, j = 2, 3, \dots, n \\ \mathbf{D}_i^T + \sum_z {}^z\mathbf{E}_{ji}^T & j = 1, i = 2, 3, \dots, n \\ \sum_z {}^z\mathbf{E}_{ij} & \text{else} \end{cases} \quad (28)$$

Each non-zero component matrix \mathbf{K}_{ij} can be considered as a pseudo stiffness component embedded between links i and j . Change of the relative angular displacement $\theta_{ij} = \cos^{-1}(\mathbf{Q}_i^T \mathbf{Q}_j)$ between links i and j induces a variation of potential energy ($\mathbf{Q}_i^T \mathbf{K}_{ij} \mathbf{Q}_j / 2$) to the system. Therefore, for a statically balanced system, all stiffness components between any two distinct links should be zeros, i.e., any off-diagonal component matrix \mathbf{K}_{ij} for $i \neq j$ in \mathbf{K} is a zero matrix as

$$\mathbf{D}_j + \sum_z {}^z\mathbf{E}_{1j} = \mathbf{0} \quad j = 2, 3, \dots, n \quad (29a)$$

$$\sum_z {}^z\mathbf{E}_{ij} = \mathbf{0} \quad i \neq 1, j \neq 1 \quad (29b)$$

By the use of Eqs. (29a) and (29b), Eq. (26) becomes constant, configuration independent and equals to

$$U = \frac{1}{2} \text{tr}(\mathcal{K}) \quad (30)$$

According to the principle of static equilibrium [1], when the total potential energy is invariant with respect to the configuration change of the attached links, the static equilibrium is achieved at any configuration of a mechanism.

Based on Eqs. (26) and (30), it is obvious that $\mathcal{Q}^T \mathcal{K} \mathcal{Q} = \text{tr}(\mathcal{K})$. For \mathcal{K} is a diagonal block matrix, there exists a square root matrix \mathcal{S} such that $\mathcal{K} = \mathcal{S}^T \mathcal{S}$. Thus, $\mathcal{Q}^T \mathcal{K} \mathcal{Q} = \mathcal{Q}^T \mathcal{S}^T \mathcal{S} \mathcal{Q}$. Let $\mathcal{X} = \mathcal{S} \mathcal{Q}$ represent a vector of an n -dimensional space. Hence, it can be obtained that $\mathcal{X}^T \mathcal{X} = \text{tr}(\mathcal{K})$. In particular for a 3-D space, it plots an isotropic sphere of radius $[\text{tr}(\mathcal{K})]^{1/2}$. Hence, the condition of Eqs. (29a) and (29b) is referred to as an isotropic condition for stiffness block matrix, i.e., the balancing condition.

It is also seen that, the isotropy may be contributed by the gravity-to-spring balancing as described in Eq. (29a), and the spring-to-spring balancing [23] as depicted in Eq. (29b). Eq. (29a) involves both the gravitational and the elastic components, while Eq. (29b) concerns only the elastic components. While gravity-to-spring balancing refers to the energy exchange between the gravitational potential energy and the elastic potential energy, spring-to-spring balancing refers to that between the elastic potential energies. Traditionally, gravity-balanced mechanisms with complex spring installations are sometimes difficult to

identify the contribution of a single spring. With the block matrix formulation, analysis can be made more efficiently by knowing the exact parameters involved in each entry of the block matrix.

By Eqs. (10), (22a) and (22b), Eq. (30) can be expressed to scalar parameters as

$$U = -m_1 g s_1 C(\sigma_1 - \gamma) - \left(\sum_{i=2}^n m_i \right) g r_1 C(\varphi_1 - \gamma) + \sum_{z=1}^N \sum_i \frac{1}{2} ({}^z c_i) \tag{31}$$

where

$${}^z c_i = \begin{cases} k_z a_z^2, & i = x \\ k_z r_i^2, & i = x + 1, x + 2, \dots, y - 1 \\ k_z b_z^2, & i = y \end{cases} \tag{32}$$

and N is the number of springs.

Eq. (31) is referred to as the pre-tensile energy in the planar manipulator. As long as each link is a rigid body, the pre-tensile energy is constant. Since the pre-tensile energy results in internal forces of a mechanism, it is should be kept as small as possible.

According to the Lagrange mechanics, the virtual work δU of an $(n - 1)$ -DOF system due to generalized forces τ_j 's with respective displacements $\delta\theta_j$'s can be expressed as

$$\delta U = \sum_{j=1}^{n-1} \tau_j \delta\theta_j \tag{33}$$

To be more specifically, quantity θ_j represents the angular displacement of an articulated joint j , for $j = 1, 2, \dots, (n - 1)$, and generalized force τ_j represents the joint torque of articulated joint j .

According to Eq. (26), the derived potential energy of the n -link, $(n - 1)$ -DOF manipulator is described in an abstract $(n - 1)$ -dimensional space constituted of the $(n - 1)$ independent vectors \mathbf{Q}_i 's. In addition to the isotropic conditions, Eq. (26) can be written as

$$U = \frac{1}{2} \sum_{i=2}^n \mathbf{Q}_i^T \mathbf{K}_{ii} \mathbf{Q}_i \tag{34}$$

Since \mathbf{K}_{ii} for $i = 2, 3, \dots, n$ is a 2×2 diagonal matrix, i.e. $\mathbf{K}_{ii} = k_{ii} \mathbf{I}$. Hence, by Eq. (34),

$$\delta U = \frac{1}{2} \sum_{i=2}^n k_{ii} \delta(\mathbf{Q}_i^T \mathbf{Q}_i) \tag{35}$$

Since,

$$\delta(\mathbf{Q}_i^T \mathbf{Q}_i) = \sum_{j=1}^{n-1} 2\mathbf{Q}_i^T \frac{\partial \mathbf{Q}_i}{\partial \theta_j} \delta\theta_j \tag{36}$$

Substituting Eq. (36) in Eq. (35) yields

$$\delta U = \sum_{j=1}^{n-1} \sum_{i=2}^n \left(k_{ii} \mathbf{Q}_i^T \frac{\partial \mathbf{Q}_i}{\partial \theta_j} \right) \delta\theta_j \tag{37}$$

Comparing Eq. (37) with Eq. (33), the joint torque of articulated joint j can be obtained as

$$\tau_j = \sum_{i=2}^n k_{ii} \mathbf{Q}_i^T \frac{\partial \mathbf{Q}_i}{\partial \theta_j} \tag{38}$$

where

$$\frac{\partial \mathbf{Q}_i}{\partial \theta_j} = \begin{cases} \mathbf{0} & \text{for } i \neq j \\ [-S\theta_j \quad C\theta_j]^T & \text{for } i = j \end{cases} \tag{39}$$

and

$$\mathbf{Q}_i^T \frac{\partial \mathbf{Q}_i}{\partial \theta_j} = 0 \tag{40}$$

Substituting Eq. (38) with Eq. (40) yields $\tau_j=0$, the isotropic condition results in a zero joint torque for each joint j . Furthermore, a force component can be observed from Eq. (38) as

$$\mathbf{F}_i^T = k_{ii}\mathbf{Q}_i^T \tag{41}$$

For $i=2, 3, \dots, n$, a force block matrix $\mathbf{F}=[\mathbf{F}_2, \mathbf{F}_3, \dots, \mathbf{F}_n]^T$ can be defined. Under the isotropic conditions, Eq. (41) can be generalized to

$$\mathcal{F} = \mathcal{K}\mathcal{Q} \tag{42}$$

According to Eq. (38), the force block matrix can also be obtained by a linear transformation by the Jacobian block matrix $\mathbf{J} = [\partial\mathbf{Q}_i / \partial\theta_j]$ as $\boldsymbol{\tau}=\mathbf{J}^T \mathcal{F}$ where $\boldsymbol{\tau}=[\tau_j]$. Hence, by defining a constant diagonal matrix as $\mathcal{D}=\mathcal{K}^T\mathcal{K}$ and based on Eq. (42), it yields

$$\mathcal{F}^T\mathcal{F} = \mathcal{Q}^T\mathcal{D}\mathcal{Q} = \text{tr}(\mathcal{D}) \tag{43}$$

Hence, it is concluded that the force component is also isotropic in an $(n - 1)$ -dimensional space. Note that, although vectors θ_i 's are physically lying on a plane, a mathematical interpretation of an $(n - 1)$ -dimensional space can be developed by the independent scalars θ_i 's.

Due to the success to integrate the formulation of gravitational potential energy with the elastic potential energy, design parameters for spring installation can be obtained from the overall stiffness block matrix. Statically balancing conditions can be achieved by maintaining the stiffness block matrix isotropic. The stiffness block matrix can also be used to examine the equilibrium for planar manipulators with linear spring installations. Note that, the stiffness block matrix is a constant if and only if the dimensional parameters of the mechanism are invariant, i.e. the distance between the centers of revolute joints must be constant. Hence, planar manipulators with only revolute joints are considered in this paper.

5. Locations of spring attachment points

In this section, one spring installation configuration is found for the balancing of a single pendulum, and two spring installation configurations are demonstrated for a double pendulum. The exact solutions for design parameters of all demonstrated spring configurations are obtained.

5.1. Spring installation for a single pendulum ($n=2$)

Consider the stiffness block matrix ${}^g\mathcal{K}$ of a single pendulum,

$${}^g\mathcal{K} = \begin{bmatrix} 2\mathbf{D}_1 & \mathbf{D}_2 \\ \mathbf{D}_2^T & \mathbf{0} \end{bmatrix} \tag{44}$$

where the off-diagonal component of interest is

$$\mathbf{D}_2 = \begin{bmatrix} -m_2gs_2C(\sigma_2-\gamma) & m_2gs_2S(\sigma_2-\gamma) \\ -m_2gs_2S(\sigma_2-\gamma) & -m_2gs_2C(\sigma_2-\gamma) \end{bmatrix} \tag{45}$$

Since \mathbf{D}_2 is the (1, 2)th component of the stiffness block matrix, a spring 1(1, 2) fitted between link 1 (ground) and link 2 is employed. The stiffness block matrix of Eq. (44) is then modified as

$$\mathcal{K} = \begin{bmatrix} 2\mathbf{D}_1 + {}^1\mathbf{E}_{11} & \mathbf{D}_2 + {}^1\mathbf{E}_{12} \\ \mathbf{D}_2^T + {}^1\mathbf{E}_{12}^T & {}^1\mathbf{E}_{22} \end{bmatrix} \tag{46}$$

where, according to Eq. (8), elastic component ${}^1\mathbf{E}_{12}$ is

$${}^1\mathbf{E}_{12} = \begin{bmatrix} -k_1a_1b_1C(\beta_1-\alpha_1) & k_1a_1b_1S(\beta_1-\alpha_1) \\ -k_1a_1b_1S(\beta_1-\alpha_1) & -k_1a_1b_1C(\beta_1-\alpha_1) \end{bmatrix} \tag{47}$$

According to Eq. (29a), letting matrix $[\mathbf{D}_2 + {}^1\mathbf{E}_{12}]$ be a zero matrix yields two effective design equations as

$$k_1a_1b_1C(\beta_1-\alpha_1) + m_2gs_2C(\sigma_2-\gamma) = 0 \tag{48a}$$

$$k_1a_1b_1S(\beta_1-\alpha_1) + m_2gs_2S(\sigma_2-\gamma) = 0 \tag{48b}$$

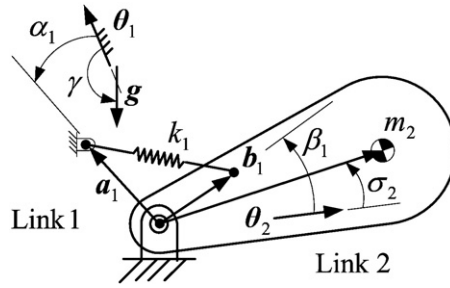


Fig. 3. A gravity-balanced one-DOF manipulator.

Summing the squared Eqs. (48a) and (48b) yields

$$k_1 a_1 b_1 = m_2 g s_2 \tag{49a}$$

Dividing of Eq. (48a) by Eq. (48b) yields

$$\beta_1 - \alpha_1 = \sigma_2 - \gamma \pm \pi \tag{49b}$$

Eq. (49a) is the well-known design equation for the spring balancing of single pendulum [1,19,20,25,26], where spring parameters k_1 , a_1 , b_1 , α_1 and β_1 can be determined. The spring installation of a single pendulum is illustrated in Fig. 3.

5.2. Spring installation for a double pendulum ($n=3$)

5.2.1. Spring configurations for a double pendulum ($n=3$)

Consider stiffness block matrix ${}^g\mathcal{K}$ of a double pendulum,

$${}^g\mathcal{K} = \begin{bmatrix} 2D_1 & D_2 & D_3 \\ & 0 & 0 \\ & & 0 \end{bmatrix} \tag{50}$$

Since all matrix components on the lower triangle of a stiffness block matrix are the transpose of the ones on the upper triangle, they are henceforth blanked for conciseness. In order to eliminate the component D_3 on the (1, 3)th entry, a spring 1(1, 3) is installed. The stiffness block matrix becomes

$${}^g\mathcal{K} + {}^{1(1,3)}\mathcal{K} = \begin{bmatrix} 2D_1 + {}^1E_{11} & D_2 + {}^1E_{12} & D_3 + {}^1E_{13} \\ & {}^1E_{22} & {}^1E_{23} \\ & & {}^1E_{33} \end{bmatrix} \tag{51}$$

Observed from Eq. (51), since the elastic component ${}^1E_{23}$ is a non-zero matrix with non-zero parameters k_1 and b_1 , a second spring is required for all off-diagonal components to be zero. In this case, the spring-to-spring balancing is applied. The two possible configurations for spring 2 are 2(2, 3) and 2(1, 3). Note that, spring 2 with configuration (1, 2) is unable to balance component ${}^1E_{23}$ on entry (2, 3). Corresponding stiffness block matrices, \mathcal{K}_I and \mathcal{K}_{II} , of the two possible configurations are expressed, respectively, as

$$\mathcal{K}_I = {}^g\mathcal{K} + {}^{1(1,3)}\mathcal{K} + {}^{2(2,3)}\mathcal{K} = \begin{bmatrix} 2D_1 + {}^1E_{11} & D_2 + {}^1E_{12} & D_3 + {}^1E_{13} \\ & {}^1E_{22} + {}^2E_{22} & {}^1E_{23} + {}^2E_{23} \\ & & {}^1E_{33} + {}^2E_{33} \end{bmatrix} \tag{52}$$

and

$$\mathcal{K}_{II} = {}^g\mathcal{K} + {}^{1(1,3)}\mathcal{K} + {}^{2(1,3)}\mathcal{K} = \begin{bmatrix} 2D_1 + {}^1E_{11} + {}^2E_{11} & D_2 + {}^1E_{12} + {}^2E_{12} & D_3 + {}^1E_{13} + {}^2E_{13} \\ & {}^1E_{22} + {}^2E_{22} & {}^1E_{23} + {}^2E_{23} \\ & & {}^1E_{33} + {}^2E_{33} \end{bmatrix} \tag{53}$$

Illustrative figures for configurations I and II are shown in Fig. 4(a) and (b), respectively. Both configurations can result in a same installation with one end of spring 2 attached to the center of the ground pivot. The spring installation with such an extreme location of spring attachment points has been proposed and investigated [27].

5.2.2. Configuration I: a mono-articular spring and a bi-articular spring

Consider the spring configuration in Fig. 4(a) where springs 1(1, 3) and 2(2, 3) are referred to as a bi-articular spring and a mono-articular spring, respectively. The “articular” is termed according to the number of articulated joints a spring is spanned over.

By letting the three off-diagonal components, \mathbf{K}_{13} , \mathbf{K}_{12} and \mathbf{K}_{23} , of \mathcal{K}_1 be zero matrices, six design equations can be obtained as

$$k_1 a_1 b_1 C(\beta_1 - \alpha_1) + d_{3,1} = 0 \tag{54a}$$

$$k_1 a_1 b_1 S(\beta_1 - \alpha_1) + d_{3,2} = 0 \tag{54b}$$

$$k_1 a_1 r_2 C(\varphi_1 - \alpha_1) + d_{2,1} = 0 \tag{54c}$$

$$k_1 a_1 r_2 S(\varphi_1 - \alpha_1) + d_{2,2} = 0 \tag{54d}$$

$$-k_1 r_2 b_1 C(\beta_1 - \varphi_2) + k_2 a_2 b_2 C(\beta_2 - \alpha_2) = 0 \tag{54e}$$

$$-k_1 r_2 b_1 S(\beta_1 - \varphi_2) + k_2 a_2 b_2 S(\beta_2 - \alpha_2) = 0 \tag{54f}$$

Let spring parameters k_1, k_2 be given parameters. Length a_1 can be first determined by the sum of squares of Eqs. (54c) and (54d) and angle α_1 can be obtained by the inverse tangent function by dividing Eq. (54d) with Eq. (54c) as

$$a_1 = \frac{\sqrt{d_{2,1}^2 + d_{2,2}^2}}{k_1 r_2} \tag{55a}$$

$$\alpha_1 = \varphi_1 - \tan^{-1}(d_{2,2} / d_{2,1}) \tag{55b}$$

Once parameters a_1 and α_1 are determined, parameters b_1 and β_1 can be obtained by Eqs. (54a) and (54b) as

$$b_1 = \frac{\sqrt{d_{3,1}^2 + d_{3,2}^2}}{k_1 a_1} \tag{56a}$$

$$\beta_1 = \alpha_1 + \tan^{-1}(d_{3,2} / d_{3,1}) \tag{56b}$$

Finally, with Eqs. (54e) and (54f),

$$a_2 b_2 = \frac{k_1 r_2 b_1}{k_2} \tag{57a}$$

$$\beta_2 - \alpha_2 = \beta_1 - \varphi_2 \tag{57b}$$

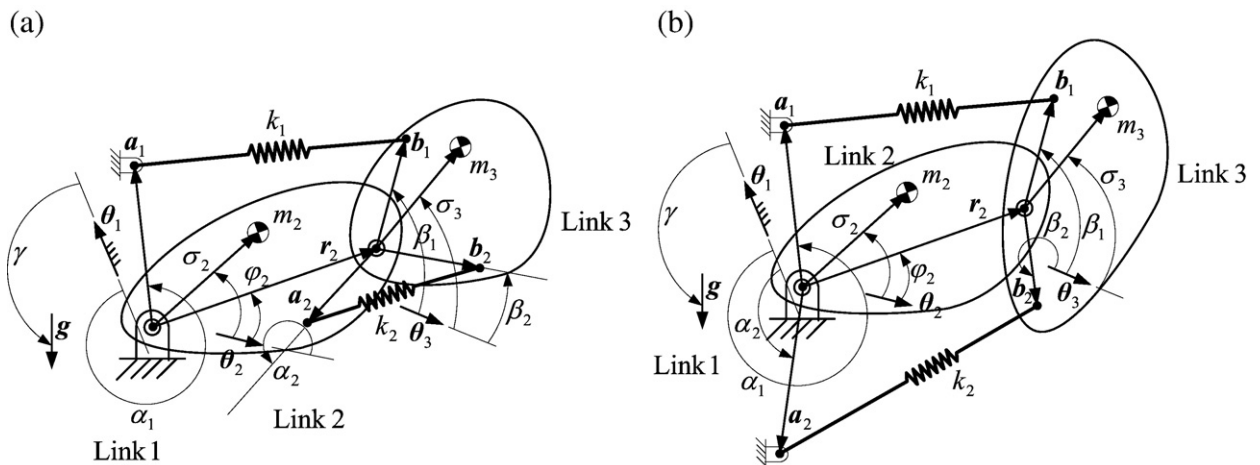


Fig. 4. The spring installation of a double pendulum with (a) a mono-articular spring and a bi-articular spring; and (b) two bi-articular springs.

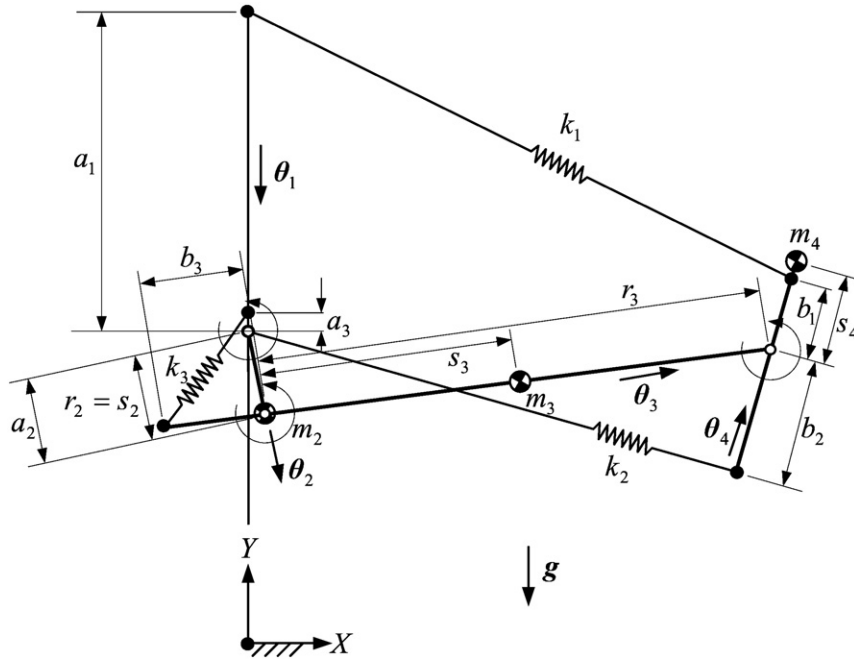


Fig. 5. A statically balanced three-DOF manipulator.

The parameters a_2 , b_2 , α_2 and β_2 on the left-hand sides of Eq. (57) can be determined.

5.2.3. Configuration II: two bi-articular springs

Consider the spring configuration in Fig. 4(b) with two bi-articular springs. By letting the three off-diagonal components, \mathbf{K}_{13} , \mathbf{K}_{12} and \mathbf{K}_{23} , of \mathcal{K}_{II} be zero matrices, six design equations can be obtained as

$$k_1 a_1 b_1 C(\beta_1 - \alpha_1) + k_2 a_2 b_2 C(\beta_2 - \alpha_2) + d_{3,1} = 0 \tag{58a}$$

$$k_1 a_1 b_1 S(\beta_1 - \alpha_1) + k_2 a_2 b_2 S(\beta_2 - \alpha_2) + d_{3,2} = 0 \tag{58b}$$

$$k_1 a_1 r_2 C(\varphi_1 - \alpha_1) + k_2 a_2 r_2 C(\varphi_2 - \alpha_2) + d_{2,1} = 0 \tag{58c}$$

$$k_1 a_1 r_2 S(\varphi_1 - \alpha_1) + k_2 a_2 r_2 S(\varphi_2 - \alpha_2) + d_{2,2} = 0 \tag{58d}$$

$$k_1 r_2 b_1 C(\beta_1 - \varphi_2) + k_2 r_2 b_2 C(\beta_2 - \varphi_2) = 0 \tag{58e}$$

$$k_1 r_2 b_1 S(\beta_1 - \varphi_2) + k_2 r_2 b_2 S(\beta_2 - \varphi_2) = 0 \tag{58f}$$

Let spring parameters k_1 , k_2 , a_2 and α_2 be given parameters. Summing the squared Eqs. (58c) and (58d) yields

$$(k_1 a_1 r_2)^2 = [k_2 a_2 r_2 C(\varphi_2 - \alpha_2) + d_{2,1}]^2 + [k_2 a_2 r_2 S(\varphi_2 - \alpha_2) + d_{2,2}]^2 \tag{59}$$

Length a_1 can be first determined by the given parameters on the right-hand side of Eq. (59). Substituting parameter a_1 in Eqs. (58c) and (58d) yields angle α_1 . Then, summing the squared Eqs. (58e) and (58f) yields the relations for parameters b_1 , b_2 , β_1 and β_2

$$k_2 b_2 = k_1 b_1 \tag{60a}$$

Table 1
Dimensional and inertia parameters of the three-DOF manipulator (data given in kg, m).

Link i	m_i	s_i	r_i
2	4.55	.127	.127
3	11.42	.378	.757
4	42.60	.140	
$\varphi_i = \sigma_i = \gamma = 0$			

Table 2
Spring design parameters of the three-DOF manipulator (data given in N/m, m, degree).

Spring i	k_i	a_i	α_i	b_i	β_i
1	1000	.491	180	.119	0
2	600	.127	180	.198	180
3	8000	.010	180	.151	180

$$\beta_2 = \beta_1 \pm \pi \tag{60b}$$

Using Eqs. (60a) and (60b) and replacing parameters b_2 and β_2 in Eqs. (58a) and (58b) with parameters b_1 and β_1 yield

$$k_1 a_1 b_1 C(\beta_1 - \alpha_1) - k_1 a_2 b_1 C(\beta_1 - \alpha_2) + d_{3,1} = 0 \tag{61a}$$

$$k_1 a_1 b_1 S(\beta_1 - \alpha_1) - k_1 a_2 b_1 S(\beta_1 - \alpha_2) + d_{3,2} = 0 \tag{61b}$$

Applying the sum and difference formulas for the sine and cosine functions in Eqs. (61a) and (61b) yields

$$A_{11}(b_1 C\beta_1) + A_{12}(b_1 S\beta_1) = -d_{3,1} \tag{62a}$$

$$A_{21}(b_1 C\beta_1) + A_{22}(b_1 S\beta_1) = -d_{3,2} \tag{62b}$$

where

$$A_{11} = A_{22} = k_1(a_1 C\alpha_1 - a_2 C\alpha_2) \tag{63a}$$

$$A_{12} = -A_{21} = k_1(a_1 S\alpha_1 - a_2 S\alpha_2) \tag{63b}$$

Since coefficients A_{11} , A_{12} , A_{21} and A_{22} are constituted of determined parameters, Eqs. (62a) and (62b) can be solved by the Cramer's rule as

$$b_1 C\beta_1 = \frac{\Delta_1}{\Delta} \tag{64a}$$

$$b_1 S\beta_1 = \frac{\Delta_2}{\Delta} \tag{64b}$$

where

$$\Delta = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \tag{65a}$$

$$\Delta_1 = \begin{vmatrix} -d_{3,1} & A_{12} \\ -d_{3,2} & A_{22} \end{vmatrix} \tag{65b}$$

$$\Delta_2 = \begin{vmatrix} A_{11} & -d_{3,1} \\ A_{21} & -d_{3,2} \end{vmatrix} \tag{65c}$$

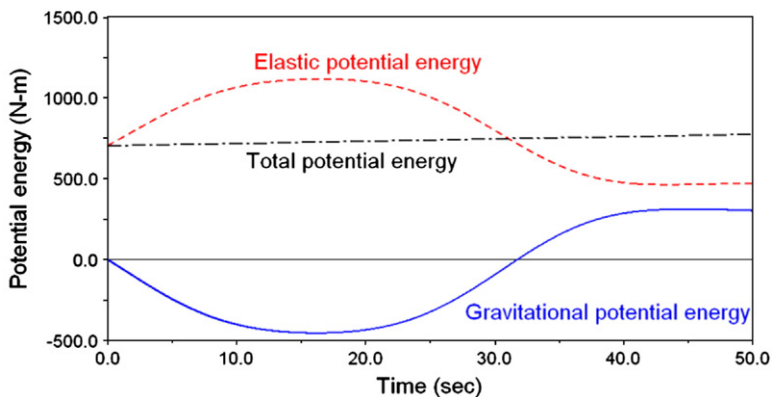
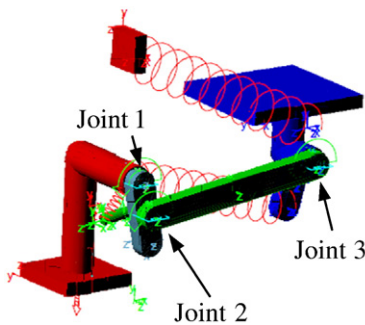


Fig. 6. A simulation model of the three-DOF manipulator.

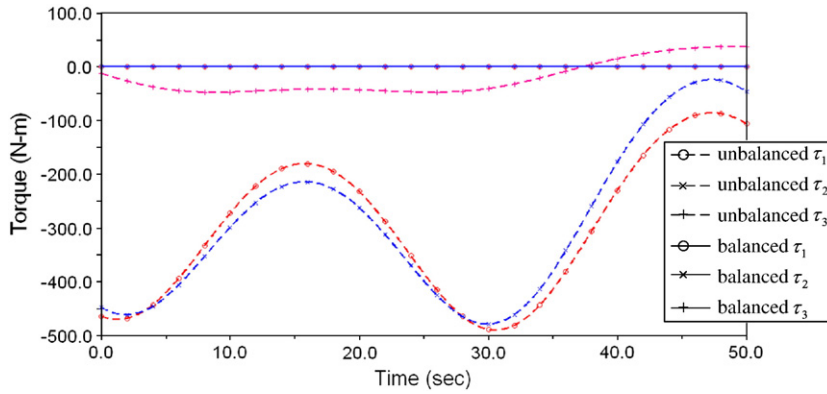


Fig. 7. The input torques of the three-DOF manipulator before and after static balance.

Length b_1 can be derived by summing the squared Eqs. (64a) and (64b), and angle β_1 can be obtained by the inverse tangent function from dividing Eq. (64b) with Eq. (64a) as

$$b_1 = \frac{\sqrt{\Delta_1^2 + \Delta_2^2}}{\Delta} \tag{66a}$$

$$\beta_1 = \tan^{-1}(\Delta_2 / \Delta_1) \tag{66b}$$

Parameters b_2 and β_2 can be solved by Eqs. (58a) and (58b) with the substitutions of determined parameters.

In the current studies of the auxiliary link method, two general arrangements of the auxiliary links were proposed; one is familiar as a structure of serially connected parallelogram linkages wherein each parallelogram is installed with a single spring [18,19]; the other one uses the parallel auxiliary links to identify the overall mass center position of the mechanism, and one of the installed springs must be fitted between the base and the mass center [28,29], this arrangement may require a larger operating space for the spring pending on the position of the masse center. In comparison to both arrangements, the design without auxiliary links in the paper can disregard the motion interferences between the auxiliary links and the primary links, and avoid any of the singularity problems which may occur, and thus, simplify the design process. However, using the multi-articular springs though may force these springs traversing a potentially large workspace. By selecting springs with appropriate stiffness, it may yield a design with spring attachment points close to the rotation centers of articular joint, i.e. short lengths a_j and b_j . Consequently, the operating space of the multi-articular springs can be minified.

6. An isotropic three-DOF articulated manipulator

The design example demonstrated in this section is a three-DOF ($n=4$) articulated manipulator. A stiffness block matrix \mathcal{K} with 4×4 component matrices and six off-diagonal components on its upper triangle is considered. A total number of 2×6 design equations can be derived. Since a spring z can provide four design parameters, namely a_z , α_z , b_z and β_z , to the design equations, three installed springs provide with a total number of twelve design parameters. For links 1, 2, 3 and 4 of the manipulator, springs 1, 2 and 3 are installed with unrepeated spring configurations 1(1, 4), 2(2, 4) and 3(1, 3), respectively. Apply the sum and difference formulas for the sine and cosine functions to the twelve design equations, and define twelve auxiliary variables X_i 's for $i = 1, 2, \dots, 12$ as

$$\begin{aligned} X_1 &= a_1 C(\alpha_1), X_2 = a_1 S(\alpha_1), X_3 = b_1 C(\beta_1), X_4 = b_1 S(\beta_1), \\ X_5 &= a_2 C(\alpha_2), X_6 = a_2 S(\alpha_2), X_7 = b_2 C(\beta_2), X_8 = b_2 S(\beta_2), \\ X_9 &= a_3 C(\alpha_3), X_{10} = a_3 S(\alpha_3), X_{11} = b_3 C(\beta_3), X_{12} = b_3 S(\beta_3) \end{aligned} \tag{67}$$

The twelve design equations can be represented by

$$f_j(X_1, X_2, \dots, X_{12}) = 0 \quad j = 1, 2, \dots, 12 \tag{68}$$

For a given three-DOF articulated manipulator as shown in Fig. 5, the inertia and dimensional parameters of the manipulator are listed in Table 1. In addition to the arbitrarily selected k_1 , k_2 and k_3 , the numerical solutions of X_i 's in Eq. (68) can be simply solved with the "NSolve" function in software Mathematica. The resultant design parameters are tabulated in Table 2. The exact solutions of Eq. (68) are described in details in Appendix A, and they are consistent with the numerical solutions.

A simulation model is built in ADAMS as shown in Fig. 6. The gravitational potential energy and the elastic potential energy are plotted in Fig. 6 by an arbitrarily given task of the manipulator. The total potential energy remains constant at any configuration.

The static force on the end effector is zero in all directions on the working plane. Fig. 7 illustrates the comparison of the static input torques with and without springs, where the dashed and solid curves respectively represent the input torques of joints 1, 2 and 3, i.e. τ_1 , τ_2 and τ_3 , before and after statically balancing. It is observed that, all input torques are greatly reduced since large gravity forces are counterbalanced, and the representative curves are all very close to the zero line in Fig. 7.

6. Conclusions

In this study, a methodology is developed to determine the spring installation for the design of a statically balanced planar articulated manipulator without parallel auxiliary links. In comparison to the auxiliary link method, the design without auxiliary links in the paper can disregard the motion interferences between the auxiliary links and the primary links, and avoid any of the singularity problems which may occur, and thus, simplify the design process. The methodology proposed in this paper can be generally applied for planar articulated manipulators and hence provide an alternative, and also attractive, design approach. Two major matrices are proposed, namely the configuration block matrix and the stiffness block matrix. By the stiffness block matrix representation, spring configurations can be determined through the placements of elastic and gravitational components in the block matrix. An isotropic stiffness block matrix provides the design equations for the determination of spring attachment points. Exact solutions of the design equations are given with details in the examples of a spring-loaded one-, two- and three-DOF articulated manipulators, in which, the spring installation of the three-DOF gravity-balanced manipulator is novel. Simulation results showed that the static equilibrium is achieved at any reachable region of the manipulators.

Appendix A

The twelve design equations of Eq. (68) can be written as

$$k_1 a_1 b_1 f(\beta_1 - \alpha_1) + d_{4,(f)} = 0 \quad (\text{A.1})$$

$$k_1 a_1 r_3 f(\varphi_3 - \alpha_1) + k_3 a_3 b_3 f(\beta_3 - \alpha_3) + d_{3,(f)} = 0 \quad (\text{A.2})$$

$$k_1 a_1 r_2 f(\varphi_2 - \alpha_1) + k_3 a_3 r_2 f(\phi_2 - \alpha_3) + d_{2,(f)} = 0 \quad (\text{A.3})$$

$$-k_1 r_2 b_1 f(\beta_1 - \varphi_2) + k_2 a_2 b_2 f(\beta_2 - \alpha_2) = 0 \quad (\text{A.4})$$

$$-k_1 r_2 r_3 f(\varphi_3 - \varphi_2) + k_2 a_2 r_3 f(\varphi_3 - \alpha_2) - k_3 r_2 b_3 f(\beta_3 - \phi_2) = 0 \quad (\text{A.5})$$

$$k_1 r_3 b_1 f(\beta_1 - \varphi_3) + k_2 r_3 b_2 f(\beta_2 - \varphi_3) = 0 \quad (\text{A.6})$$

where f^* represents C^* or S^* , and correspondingly $d_{i,(f)}$ is $d_{i,1}$ (or $d_{i,2}$) when f represents a cosine (or a sine) function.

Eq. (A.4) yields

$$k_1 r_2 b_1 = k_2 a_2 b_2 \quad (\text{A.7})$$

$$\beta_1 - \varphi_2 = \beta_2 - \alpha_2 \quad (\text{A.8})$$

Eq. (A.6) yields

$$k_1 r_3 b_1 = k_2 r_3 b_2 \quad (\text{A.9})$$

$$\beta_1 - \varphi_3 = \beta_2 - \varphi_3 \pm \pi \quad (\text{A.10})$$

Dividing Eq. (A.7) with Eq. (A.9) yields

$$a_2 = r_2 \quad (\text{A.11})$$

Subtracting Eq. (A.10) from Eq. (A.8) gives

$$\alpha_2 = \varphi_2 \pm \pi \quad (\text{A.12})$$

Substituting Eqs. (A.11) and (A.12) into Eq. (A.5) yields

$$(k_1 + k_2) r_2 r_3 f(\varphi_3 - \varphi_2) + k_3 r_2 b_3 f(\beta_3 - \varphi_2) = 0 \quad (\text{A.13})$$

Hence, parameters b_3 and β_3 can be determined as

$$b_3 = (k_1 + k_2) r_3 / k_3 \quad (\text{A.14})$$

$$\beta_3 = \phi_3 \pm \pi \quad (\text{A.15})$$

Let $Y_1 = a_1C(\alpha_1)$, $Y_2 = a_1S(\alpha_1)$, $Y_3 = a_3C(\alpha_3)$ and $Y_4 = a_3S(\alpha_3)$. Applying the sum and difference formulas for the sine and cosine functions in Eqs. (A.2) and (A.3) yields a linear system with four equations as

$$\sum_{i=1}^4 A_{ij}Y_i = B_j, \quad j = 1, 2, 3, 4 \quad (\text{A.16})$$

where A_{ij} and B_j are constituted of determined parameters.

With the Cramer's rule, auxiliary variables Y_1 , Y_2 , Y_3 and Y_4 can be solved. Parameters a_1 , α_1 , a_3 and α_3 can be also determined accordingly. Finally, with Eq. (A.1), parameters b_1 and β_1 can be solved as

$$b_1 = \frac{\sqrt{d_{4,1}^2 + d_{4,2}^2}}{k_1 a_1} \quad (\text{A.17})$$

$$\beta_1 = \alpha_1 + \tan^{-1}(d_{4,2} / d_{4,1}) \quad (\text{A.18})$$

Substitute Eqs. (A.17) and (A.18) into Eqs. (A.9) and (A.10), respectively. Parameters b_2 and β_2 can be obtained.

References

- [1] C.M. Gosselin, Adaptive robotic mechanical systems: a design paradigm, *ASME Journal of Mechanical Design* Vol. 128 (1) (2006) 192–198.
- [2] K.J. Fisher, Counterbalance mechanism positions a light with surgical precision, *Mechanical Engineering* 114 (5) (1992) 76–80.
- [3] Tuda, G., Mizuguchi, O., 1983, "Arm with gravity-balancing function", United States Patent No. 4383455.
- [4] Haven, K. R., 1987, "Test system manipulator arm", United States Patent No. 4695024.
- [5] Dayton, D. C., Ardito, J. D., 1992, "Ergonomic equipment arm", United States Patent No. 5092552.
- [6] Copeland, S., McAllister, M., 1997, "Locking universal support arm", United States Patent No. 5683064.
- [7] Holung, J. A., 2005, "Mechanical arm and counterbalance assembly", United States Patent No. 6921056B2.
- [8] Blumenkranz, S. J., 2005, "Ceiling and floor mounting surgical robot set-up arms", United States Patent No. 6933695B2.
- [9] Nagasaka, M., 1976, "Sustaining device", United States Patent No. 3973748.
- [10] Krogsrud, J. C., 1979, "Counterbalance arm", United States Patent No. 4160536.
- [11] Aaldenberg, E. R., Soporowski, A. L., 1991, "Convertible swing arm and desk top copy holder", United States Patent No. 4987690.
- [12] Sander, U., 2003, "Stand arrangement", United States Patent No. 6543914B2.
- [13] Brown, G., DiGuilio, A. O., 1980, "Support apparatus", United States Patent No. 4208028.
- [14] M. Carricato, C. Gosselin, A statically balanced Gough/Stewart-type platform: conception, design and simulation, *ASME Journal of Mechanisms and Robotics* 1 (2009) 031005.
- [15] R. Rizk, S. Krut, E. Dombre, Design of a 3D gravity balanced orthosis for upper limb, 2008 IEEE International Conference on Robotics and Automation, Pasadena, CA, USA, May 19–23, 2008, pp. 2447–2452.
- [16] S. Lessard, I. Bonev, P. Bigras, S. Briot, V. Arakelian, Optimum static balancing of the parallel robot for medical 3D-ultrasound imaging, *Proceedings of the 12th World Congress in Mechanism and Machine Science*, Jun. 18–21, Besançon, France, 2007, Vol. 5, 2007, pp. 321–326.
- [17] M.J. French, M.B. Widdien, The spring-and-lever balancing mechanism, George Carwardin and the anglepoise lamp, *Proceedings of the Institution of Mechanical Engineers* 214 (2000) 501–508 Part C.
- [18] R.H. Nathan, A constant force generation mechanism, *ASME Journal of Mechanisms, Transmissions, and Automation in Design* 107 (12) (1985) 508–512.
- [19] T. Rahman, R. Ramanathan, R. Seliktar, W. Harwin, A simple technique to passively gravity-balance articulated mechanisms, *ASME Journal of Mechanical Design* 117 (4) (1995) 655–658.
- [20] D.A. Streit, E. Shin, Equilibrators for planar linkages, *ASME Journal of Mechanical Design* 115 (3) (1993) 604–611.
- [21] W.D. Dorsser, R. Barents, B.M. Wisse, J.L. Herder, Gravity-balanced arm support with energy-free adjustment, *ASME Journal of Medical Device* 1 (2007) 151–158.
- [22] S.K. Agrawal, A. Fattah, Theory and design of an orthotic device for full or partial gravity-balancing of a human leg during motion, *IEEE Transactions on Neural Systems and Rehabilitation Engineering* 12 (3) (2004) 157–165.
- [23] C. Gosselin, Stiffness mapping for parallel manipulators, *IEEE Transactions on Robotics and Automation* 6 (3) (1990) 377–382.
- [24] Tsai, L.-W., 2000, *Mechanism Design-Enumeration of Kinematic Structures According to Function*, CRC Press Technology & Industrial, pp. 261, 280–281, 289, 292.
- [25] R. Barents, M. Schenk, W.D. von Dorsser, B.M. Wisse, J.L. Herder, Spring-to-spring balancing as energy-free adjustment method in gravity equilibrators, *Proceedings of the ASME 2009 International Design Engineering Technical Conference*, Aug. 30–Sep. 2, San Diego, CA., DETC2009-86770, 2009.
- [26] P.-Y. Lin, W.-B. Shieh, D.-Z. Chen, Design of perfectly static-balanced one-DOF planar linkage with revolute joint only, *ASME Journal of Mechanical Design* 131 (5) (2009) 051004.
- [27] V. Arakelian, S. Ghazaryan, Improvement of balancing accuracy of robotic systems: application to leg orthosis for rehabilitation devices, *Mechanism and Machine Theory* 43 (5) (2008) 565–575.
- [28] S.K. Agrawal, G. Gardner, S. Pledgie, Design and fabrication of an active gravity balanced planar mechanism using auxiliary parallelograms, *ASME Journal of Mechanical Design* 123 (4) (2001) 525–528.
- [29] S.K. Agrawal, A. Fattah, Gravity-balancing of spatial robotic manipulators, *Mechanism and Machine Theory* 39 (2004) 1331–1344.