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Determination of spring installation configuration on statically balanced planar articulated manipulators

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ABSTRACT

This paper presents a design methodology to determine spring configuration on statically balanced planar articulated manipulator. From the energy perspective, the summation of potential energies remains constant at any configuration. The gravitational potential energy changes due to the movement of linkages, and the elastic potential energy corresponds to the spring configuration. By formulating similar representation of matrix form, the equilibrium equation is simplified as the summation of gravitational stiffness block matrix and elastic stiffness block matrices remain unchanged. This paper discusses the distribution features of entries in gravitational stiffness block matrix, the characteristic of elastic stiffness block matrix associated with the attachment angles and attachment points of the spring, and the equivalent spring installations that produce same elastic potential effect but with different design parameters or configurations. According to the interrelation between gravitational and elastic stiffness block matrix, the general criteria of the admissible spring configuration are obtained. For minimum number of springs and minimum total number of articulated joints that springs span over, the additional criteria and admissible spring configurations are derived. A three-degree-of-freedom planar articulated manipulator is shown as an illustrative example.

1. Introduction

Statically balanced system has been much debated in the last few decades. A mechanism is statically balanced if it maintains static equilibrium at any configuration without actuator working against gravity. Therefore, statically balanced mechanism is efficient and easily controlled. It is consequently expensively applied to existing systems, such as dentist light, equipoised lamp, excavator, crane, prosthesis [1] and orthoses in human rehabilitation [2,3].

Over the years, several methods have been proposed to achieve static balance. The counterweight method and the spring-balancing method are two of the most commonly employed. Counterweight method [4], which is simple and directly perceived through senses, utilizes additional inertia to counterbalance the gravitation. Nonetheless, counterweight method is not practical as the heavier the mechanism, the lager amount of additional inertia has to be added. To maintain identical amount of input torque [5], the spring-balancing method obtains academic values using springs, with little of additional inertia, to absorb the change in the gravitational potential energy due to the movement of linkages. That is to say, the total conservative energy remains constant, and the gravitational potential energy (U^g) and elastic potential energy (U^s) can be entirely transferred at any configuration, which can be represented as Eq. (1).

$$U^g + U^s = U = constant. \tag{1}$$

One of the approaches for spring-balancing methods is to install spring with auxiliary mechanism. Previously, auxiliary linkages are used to make sure that the bar with springs attached remains vertical, so that each link, which corresponds to a four-bar parallelogram,

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can be regarded as a simplest 1-DOF manipulator. For an n-links manipulator with $(n-1)$-DOF, general solution is then derived [6,7]. While the auxiliary parallelograms are applied to locate the center of mass, the springs are attached to center of mass in order to keep spatial mechanism in static balance [8,9]. However, when adding auxiliary linkages, there may be the interference of linkages and problem of mechanical tolerance in manufacturing. Hence, another approach, direct spring installation, has consequently become popular [3,10–18]. Although direct spring installation has been developed for several years and have already applied to practical devices [14,15], most research focused on specific mechanisms to achieve their objectives. Rather little has been studied on the determination of spring configuration. A feasible spring configuration of 1-DOF planar linkages [16] and minimum number of spring planar articulated manipulator [17] have been proposed by Lin et al.

This paper examines spring installation configuration from the interrelations between elastic potential effect of spring and gravitational potential effect of manipulator. The contributions of installation type of spring, attachment angles and attachment points of the spring installation to the static balancing are explored. The general criteria for determining feasible spring configuration of n-link statically balanced planar articulated manipulator are proposed accordingly. With such characteristics of elastic potential effect of spring and the general criteria for spring configuration determination, the additional criteria and the admissible configuration matrix with respect to an arbitrary design objective can be easily derived.

The proposed method in this paper is based on the stiffness matrix approach [17,18]. However, the coordinate system is redefined, and the stiffness block matrix is amended to be same unit as the spring constant, without multiplying lengths of link, which is physically identical as Hooke's law. In addition, this paper shows more possibilities of spring installation configuration, since the spring constant k is not predetermined and the number of springs fitted between each two links is not limited to less than or equal one. Without these two constraints, the number of admissible configuration matrix with minimum spring number generated in this paper is much more than that of proposed in [17].

The structure of the paper is as follows. Sections 2 and 3 derive the formulation of the elastic potential energy and gravitational potential energy represented by the stiffness block matrix respectively. Within the same matrix form of representation, in Section 4, based on the statically balanced condition, the equilibrium of conservative energy, as shown in Eq. (1), is simplified as the summation of gravitational stiffness block matrix and elastic stiffness block matrices, which is equal to a constant stiffness block matrix. In Section 5, in order to figure out the interrelations between gravitational stiffness block matrix and elastic stiffness block matrix, the characteristics of elastic stiffness block matrix according to the type of spring, attachment angles, and the attachment points are explored; equivalent spring installations that produce same elastic potential effect are discussed. Section 6 develops the general criteria of the admissible spring configuration for the design of an n-link statically balanced planar articulated manipulator. In Section 7, the additional criteria and the admissible configuration matrix for two specific objective functions are determined; one minimizes the number of springs and the other minimizes the number of articulated joints which springs span over. Section 8 shows illustrative examples of a three-DOF statically balanced planar articulated manipulator with both design objectives. The conclusion is drawn in Section 9.

2. Elastic stiffness block matrix representation of elastic potential energy

The coordinate system of an n-link planar articulated manipulator is defined, as shown in Fig. 1, which is different from that of suggested by Lin et al. [18]. The Denavit–Hartenberg representation [19] is applied. The Cartesian planar coordinate frame is supposed to each local coordinate system, hence the unit axes x_j and y_j are orthonormal vectors. Each succeeding coordinate

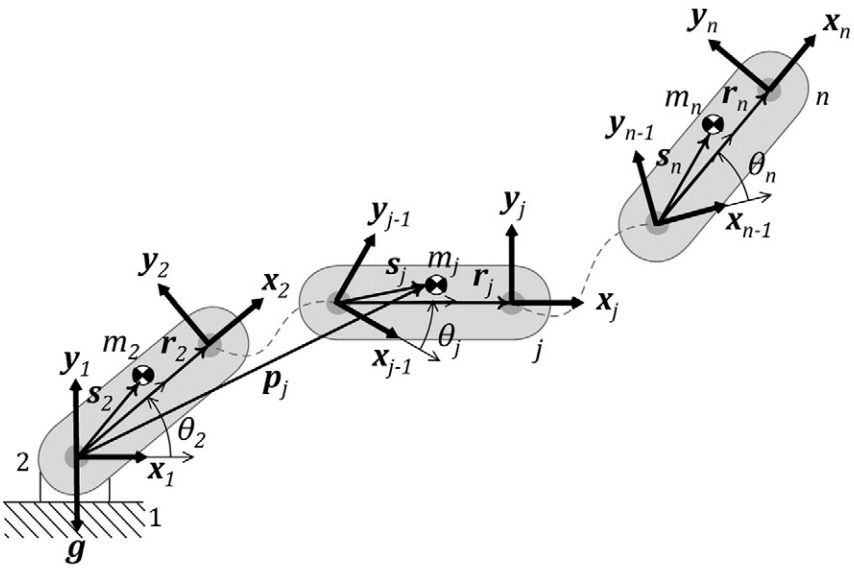


Fig. 1. Coordinate systems of an n-link planar articulated manipulator.

system with respect to its preceding coordinate system can be represented by a 2 × 2 rotation matrix $\boldsymbol{R}(\theta)$, where $\theta = \theta j$ is the joint angle from the x_{j-1} axis to the x_j axis. Therefore, once a link rotates, the coordinate systems of the link and all succeeding links would change accordingly.

For a statically balanced planar articulated manipulator, a spring configuration matrix $\boldsymbol{\Lambda}$, a $n \times n$ matrix, is defined to represent the configuration of spring installation [17]. $\boldsymbol{\Lambda} = [\lambda_{uv}]$, where element λ_{uv} lies in the row u, column v stands for the number of springs fitted between links u and v. Due to symmetry and redundancy, only the upper triangular spring configuration matrix is considered.

As shown in Fig. 2, a spring with spring constant k^{ik}, fitted in between links i and k of an n-link manipulator is denoted by $\lambda_{ik} = 1$. $\boldsymbol{a}^{ik}$ and $\boldsymbol{b}^{ik}$ are vectors from the joints of links i and k respectively to the attachment points of the spring, which can be expressed as:

$$\boldsymbol{a}^{ik} = \mathbf{A}^{ik}\boldsymbol{r}_i = \frac{a^{ik}}{r_i}\boldsymbol{R}\left(\alpha^{ik}\right)\ \boldsymbol{r}_i \tag{2}$$

$$\boldsymbol{b}^{ik} = \boldsymbol{B}^{ik}\boldsymbol{r}_k = \frac{b^{ik}}{r_k}\boldsymbol{R}\left(\beta^{ik}\right)\ \boldsymbol{r}_k. \tag{3}$$

$\boldsymbol{A}^{ik}$ and $\boldsymbol{B}^{ik}$ are the transformation matrices of $\boldsymbol{a}^{ik}$ and $\boldsymbol{b}^{ik}$ respectively. $\boldsymbol{r}j$ is the direction vector of link j. $\mathbf{R}(\alpha^{ik})$ and $\mathbf{R}(\beta^{ik})$ are 2 × 2 rotation matrices, α^{ik} and β^{ik} are the attachment angles from the line passing through joint of link i to vector $\boldsymbol{a}^{ik}$ and from the line passing through joint of link k to vector $\boldsymbol{b}^{ik}$ respectively. Further, it is defined that both $\boldsymbol{a}^{ik}$ and $\boldsymbol{b}^{ik}$ cannot be equal to zero. Since the zero-free-length springs are utilized, the distance between two attachment points $|\mathbf{x}^{ik}|$ can be regarded as the elongation of spring [18].

By expressing each vector as the form of a transformation matrix times a direction vector as Eqs. (2) and (3), the elastic potential energy is derived as:

$$U^{ik} = \frac{1}{2}k^{ik}\boldsymbol{x}^{ikT}\boldsymbol{x}^{ik} \tag{4}$$

which can be arranged as

$$U^{ik} = \frac{1}{2}\begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{r}_2\\ \vdots\\ \boldsymbol{r}_j\\ \vdots\\ \boldsymbol{r}_n\end{bmatrix}^T \boldsymbol{K}^{ik}\begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{r}_2\\ \vdots\\ \boldsymbol{r}_j\\ \vdots\\ \boldsymbol{r}_n\end{bmatrix} \tag{5}$$

where

$$\mathbf{K}^{ik} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\ & \mathbf{K}_{ii}^{ik} & \cdots & \mathbf{K}_{iv}^{ik} & \cdots & \mathbf{K}_{ik}^{ik} & \vdots\\ & & \ddots & & & \vdots & \vdots\\ & & & \mathbf{K}_{uv}^{ik} & & \mathbf{K}_{uv}^{ik} & \vdots\\ & & & & \ddots & \vdots & \vdots\\ & & & & & \mathbf{K}_{kk}^{ik} & \vdots\\ & & & & & & 0 \end{bmatrix}. \tag{6}$$

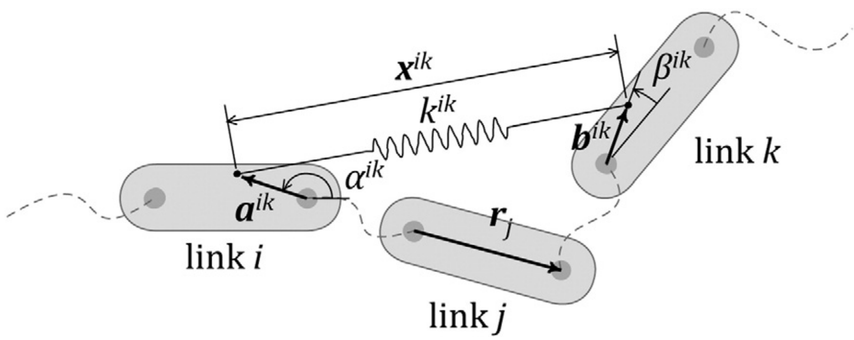


Fig. 2. A spring fitted between links i and k of a planar articulated manipulator.

$\boldsymbol{K}^{ik}$ is named the elastic stiffness block matrix. For the reason that $\mathbf{K}_{uv}^{ik}$ is the transpose of $\mathbf{K}_{vu}^{ik}$, and only the upper triangular matrix is considered as represented [18]. Entries $\mathbf{K}_{uv}^{ik}$ with $u \neq v$ in off-diagonal $\boldsymbol{K}^{ik}$,

$$\mathbf{K}_{uv}^{ik} = \begin{cases} -k^{ik}\mathbf{A}^{ik^T} = -k^{ik}\dfrac{a^{ik}}{r_i}\mathbf{R}\left(-\alpha^{ik}\right) & u = i;\ \ v = i+1,\ \ldots,\ k-1 \quad \text{(a)} \\ -k^{ik}\mathbf{A}^{ik^T}\mathbf{B}^{ik} = -k^{ik}\dfrac{a^{ik}}{r_i}\dfrac{b^{ik}}{r_k}\mathbf{R}\left(\beta^{ik}-\alpha^{ik}\right) & u = i; v = k \quad \text{(b)} \\ k^{ik}\mathbf{I} & u, v = i+1,\ \ldots,\ k-1 \quad \text{(c)} \\ k^{ik}\mathbf{B}^{ik} = k^{ik}\dfrac{b^{ik}}{r_k}\mathbf{R}\left(\beta^{ik}\right) & u = i+1,\ \ldots,\ k-1;\ \ v = k. \quad \text{(d)} \end{cases} \tag{7}$$

and entries $\mathbf{K}_{uv}^{ik}$ with $u = v$ on the diagonal of $\boldsymbol{K}^{ik}$,

$$\mathbf{K}_{uv}^{ik} = \begin{cases} k^{ik}\mathbf{A}^{ik^T}\mathbf{A}^{ik} = k^{ik}\left(\dfrac{a^{ik}}{r_i}\right)^2\mathbf{I} & u = v = 1 \quad \text{(a)} \\ k^{ik}\mathbf{I} & u = \ \ v = i+1,\ \ldots,\ k-1 \quad \text{(b)} \\ k^{ik}\mathbf{B}^{ik^T}\mathbf{B}^{ik} = k^{ik}\left(\dfrac{b^{ik}}{r_k}\right)^2\mathbf{I} & u = v = k. \quad \text{(c)} \end{cases} \tag{8}$$

where $\boldsymbol{I}$ is a 2×2 identity matrix. According to Eqs. (6), (7a–d) and (8a–c), $\boldsymbol{K}^{ik}$ can be shown as a function of spring parameters as follows,

$$\mathbf{K}^{ik} = f\left(k^{ik},\ \frac{a^{ik}}{r_i},\ \frac{b^{ik}}{r_k},\ \alpha^{ik},\ \beta^{ik},\right) = \left[\mathbf{K}_{uv}^{ik}\right]. \tag{9}$$

The elastic stiffness block matrix is the function of spring constant, the ratio of the distance from the joint to the attachment point of spring and the length of the links which the spring attaches on, and the attachment angles. Each component matrix of the elastic stiffness block matrix shows the effect of the spring between two links, which is represented by a spring constant times dimensionless distance parameters, showing more physical implication than that of proposed in the original stiffness matrix approach [18]. Notice the special case for ground-attached springs that $i = 1$

$$\mathbf{a}^{1k} = \mathbf{A}^{1k}\mathbf{x}_1 = a^{1k}\mathbf{R}\left(\alpha^{1k}\right)\mathbf{x}_1. \tag{10}$$

Hence, Eqs.(7a,b) and (8a) are varied. According to Eq. (10), since the unit vector of base coordinate system $\boldsymbol{x}_1$ is orthogonal to the gravitational acceleration vector, the representation of attachment angle α^{1k} of ground-attached springs is 90° different from that of springs that is not attached to the ground. However, the distribution feature and the characteristic of equivalent spring installation discussed in the following sections are still applicable.

3. Gravitational stiffness block matrix representation of gravitational potential energy

Referring to an n-link planar articulated manipulator as shown in Fig. 1, m_j represents the mass of link j, and the position vector of mass center of link j denoted by $\boldsymbol{p}_j$ can be expressed as:

$$\boldsymbol{p}_j = \sum\nolimits_{w=2}^{j-1}\boldsymbol{r}_w + \mathbf{s}_j \tag{11}$$

$\boldsymbol{r}_w$ is the direction vector of link w. $\boldsymbol{s}_j$ stands for the vector from the joint of link j to the mass center of link j, which can be expressed as:

$$\mathbf{s}_j = s_j\mathbf{R}\left(\delta_j\right)\mathbf{x}_j = \mathbf{D}_j\mathbf{r}_j \tag{12}$$

$$\mathbf{D}_j = \frac{s_j}{r_j}\mathbf{R}\left(\delta_j\right) \tag{13}$$

where $\mathbf{R}(\delta_j)$ is a 2×2 rotation matrix, δ_j is the angle from $\boldsymbol{x}_j$ axis to vector $\boldsymbol{s}_j$, and $\boldsymbol{D}_j$ is the transformation matrix of $\boldsymbol{s}_j$. The gravitational acceleration vector $\boldsymbol{g}$ can be represented as the form similar to Eq. (12) with $\boldsymbol{D}_1$, the transformation matrix of $\boldsymbol{g}$, as:

$$\mathbf{g} = -g\mathbf{y}_1 = \mathbf{D}_1\mathbf{x}_1 \tag{14}$$

$$\mathbf{D}_1 = g\mathbf{R}\left(\frac{3\pi}{2}\right). \tag{15}$$

By Eq. (11) and expressing vectors as the form of a transformation matrix times a direction vector as Eqs. (12) and (14), the gravitational potential energy is derived as:

$$U^g = -\sum_{j=2}^{n} m_j \boldsymbol{g}^T \boldsymbol{p}_j \tag{16}$$

which can be arranged as

$$U^g = \sum_{j=2}^{n} \boldsymbol{x}_1{}^T \boldsymbol{K}_{1j}^g \boldsymbol{r}_j \tag{17}$$

and

$$\mathbf{K}_{1j}^g = \mathbf{D}_1{}^T\left(-m_j\mathbf{D}_j - \sum_{w=j+1}^{n} m_w \mathbf{I}\right). \tag{18}$$

Since the summation term in Eq. (17) is a scalar function, the gravitational potential energy U^g shown in Eq. (17) can be further expressed [17] as

$$U^g = \frac{1}{2}\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{r}_2 \\ \vdots \\ \boldsymbol{r}_j \\ \vdots \\ \boldsymbol{r}_n \end{bmatrix}^T \boldsymbol{K}^g \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{r}_2 \\ \vdots \\ \boldsymbol{r}_j \\ \vdots \\ \boldsymbol{r}_n \end{bmatrix} \tag{19}$$

and

$$\mathbf{K}^g = \begin{bmatrix} 0 & \mathbf{K}_{12}^g & \cdots & \mathbf{K}_{1j}^g & \cdots & \mathbf{K}_{1n}^g \\ & 0 & \cdots & \cdots & \cdots & 0 \\ & & \ddots & & & \vdots \\ & & & \ddots & & \vdots \\ & & & & \ddots & \vdots \\ & & & & & 0 \end{bmatrix} \tag{20}$$

where $\mathbf{K}_{1j}^g$ is shown in Eq. (18). $\boldsymbol{K}^g$ is called the gravitational stiffness block matrix. Due to its symmetry [18], $\boldsymbol{K}^g$ can be expressed as a symmetric block matrix as shown in Eq. (20), that only the upper triangular matrix is considered. According to Eqs. (18)–(20), each component matrix $\mathbf{K}_{1j}^g$ in gravitational stiffness block matrix $\boldsymbol{K}^g$ represents the quantity of gravitational effect that acts between ground and link j. Hence, $\boldsymbol{K}^g$ has non-zero component matrices only in the first row of off-diagonal matrix as shown in Eq. (20). In addition, the magnitude of these $n - 1$ component matrices are distinct to each other in an ascending order from $j = n$ to $j = 2$. According to Eqs. (18) and (20), $\boldsymbol{K}^g$ is the function of manipulator parameters as

$$\mathbf{K}^g = f\left(m_j,\ \delta_j,\ \frac{s_j}{r_j}\right) = \left[\mathbf{K}_{1j}^g\right]. \tag{21}$$

4. Statically balanced condition

According to the law of conservative energy as shown in Eq. (1), where the summation of elastic potential energy is due to total number of N springs, the total potential energy remains unchanged regardless of the configuration of manipulator is. By formulating both gravitational potential energy and elastic potential energy as shown in Eqs. (5) and (19) respectively, the total potential energy under statically balanced condition can be expressed as

$$U=\frac{1}{2}\begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{r}_2\\ \vdots\\ \boldsymbol{r}_j\\ \vdots\\ \boldsymbol{r}_n\end{bmatrix}^T \boldsymbol{K}^g \begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{r}_2\\ \vdots\\ \boldsymbol{r}_j\\ \vdots\\ \boldsymbol{r}_n\end{bmatrix}+\sum\nolimits_N \frac{1}{2}\begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{r}_2\\ \vdots\\ \boldsymbol{r}_j\\ \vdots\\ \boldsymbol{r}_n\end{bmatrix}^T \boldsymbol{K}^{ik} \begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{r}_2\\ \vdots\\ \boldsymbol{r}_j\\ \vdots\\ \boldsymbol{r}_n\end{bmatrix}=constant. \tag{22}$$

Because of the same form of representation of both gravitational potential energy and elastic potential energy, the equation can be simplified as

$$\boldsymbol{K}^g+\sum\nolimits_N \boldsymbol{K}^{ik}=\boldsymbol{K}=[\boldsymbol{K}_{uv}] \tag{23}$$

where $\boldsymbol{K}$ is a constant stiffness block matrix and $\boldsymbol{K}_{uv}$ is the stiffness component matrix of overall system between links u and v.

A variable relative angular position $\Delta(\sum_{w=u}^{v-1}\theta_w)$ results the change in potential energy to the system corresponding to $\boldsymbol{K}_{uv}$. Hence, for a statically balanced system, the off-diagonal terms in $\boldsymbol{K}$ have to be zero. In other words, for a constant stiffness block matrix $\boldsymbol{K}$, non-zero terms only appear on the main diagonal. While discussing the spring installation configuration of statically balanced manipulator, only the off-diagonal terms of upper triangular matrix of $\boldsymbol{K}^g$ and $\boldsymbol{K}^{ik}$ are considered in comparison.

Since the manipulator parameters are known, according to Eq. (21), the gravitational potential energy is fixed. In order to satisfy the condition of static balancing, springs are installed to generate the elastic potential effect to compensate the gravitational potential effect of the manipulator. That is, the corresponding $\boldsymbol{K}^{ik}$ for each spring is designed to zero the off-diagonal terms of upper triangular matrix of $\boldsymbol{K}^g$. Thus, the spring installation configuration associated with the elastic stiffness block matrices can be determined.

Generally, the mass center of links is aligned on the line passing through the joints of links. In this paper, parameter δ_j in Eqs. (12) and (13) is assumed to be zero. Eq. (18) can be arranged as:

$$\mathbf{K}^g_{1j}=-\left(\frac{s_j}{r_j}m_j+\sum\nolimits_{w=j+1}^{n}m_w\right)g\mathbf{I}'. \tag{24}$$

All component matrices in $\boldsymbol{K}^g$ result in uniform form of a negative value times a matrix as:

$$\boldsymbol{I}'=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}. \tag{25}$$

To balance the off-diagonal matrix of $\boldsymbol{K}^g$, Eqs. (20) and (24), ground-attached springs must be installed with α^{1k} equal to 90° or 270° and β^{1k} equal to 0° or 180° according to Eqs. (6), (7a–d), (8a–c), while springs are not attached to ground, the strings have to be embedded with α^{ik} and β^{ik} equal to 0° or 180°. According to Eq. (7a–d), for ground-attached spring with $i=1$, entries $\mathbf{K}^{1k}_{uv}$ in the off-diagonal matrix of $\boldsymbol{K}^{1k}$ can be arranged as:

$$\mathbf{K}^{1k}_{uv}=\begin{cases}-k^{1k}\mathbf{A}^{1k^T}=-k^{1k}a^{1k}\sin\left(-\alpha^{1k}\right)\ \mathbf{I}' & u=1;\ v=2,\ \dots,\ k-1 \quad \text{(a)}\\ -k^{1k}\mathbf{A}^{1k^T}\mathbf{B}^{1k}=-k^{1k}a^{1k}\dfrac{b^{1k}}{r_k}\sin\left(\beta^{1k}-\alpha^{1k}\right)\ \mathbf{I}' & u=1;\ v=k \quad \text{(b)}\\ k^{1k}\mathbf{I} & u,\ v=2,\ \dots,\ k-1 \quad \text{(c)}\\ k^{1k}\mathbf{B}^{1k}=k^{1k}\dfrac{b^{1k}}{r_k}\cos\left(\beta^{1k}\right)\ \mathbf{I} & u=2,\ \dots,\ k-1;\ v=k. \quad \text{(d)}\end{cases} \tag{26}$$

For springs not attached to the ground with $i\neq 1$, entries $\mathbf{K}^{ik}_{uv}$ in the off-diagonal matrix of $\boldsymbol{K}^{ik}$ are shown as:

$$\mathbf{K}^{ik}_{uv}=\begin{cases}-k^{ik}\mathbf{A}^{ik^T}=-k^{ik}a^{ik}\cos\left(-\alpha^{ik}\right)\ \mathbf{I} & u=i;\ v=i+1,\ \dots,\ k-1 \quad \text{(a)}\\ -k^{ik}\mathbf{A}^{ik^T}\mathbf{B}^{ik}=-k^{ik}a^{ik}\dfrac{b^{ik}}{r_k}\cos\left(\beta^{ik}-\alpha^{ik}\right)\ \mathbf{I} & u=i;\ v=k \quad \text{(b)}\\ k^{ik}\mathbf{I} & u,\ v=i+1,\ \dots,\ k-1 \quad \text{(c)}\\ k^{ik}\mathbf{B}^{ik}=k^{ik}\dfrac{b^{ik}}{r_k}\cos\left(\beta^{ik}\right)\ \mathbf{I} & u=i+1,\ \dots,\ k-1;\ v=k. \quad \text{(d)}\end{cases} \tag{27}$$

Likewise, all component matrices in $\boldsymbol{K}^{ik}$ result in a value times a matrix I' or I. Hence, only the value of each component matrix has to be considered while comparing the off-diagonal terms of upper triangular matrix of $\boldsymbol{K}^{g}$ and $\boldsymbol{K}^{ik}$ under statically balanced condition.

5. Elastic potential effect of spring

Topics on design parameters, such as the attachment angles and attachment points of the spring installation, influence the elastic potential effect, and the equivalent spring installations that produce same elastic potential effect that are discussed in the following paragraphs.

5.1. Distribution feature of entries in elastic stiffness block matrix

A spring attached to a set of unconnected links that spans over multiple articulated joints is called a multi-articular spring. Among which, springs spanning over two articulated joints are bi-articular springs; springs spanning over three articulated joints are tri-articular springs, and so forth. Through Eqs. (6), (7a–d), and (8a–c), each component matrix represents the elastic potential between each two links. Attaching a multi-articular spring between links i and k, corresponding to λ_{ik} in $\boldsymbol{\Lambda}$ equals to 1, causes elastic potential effect between links i and k. Hence, the non-zero component matrices are contributed in $\boldsymbol{K}^{ik}$ from the row i to row k and from column i to column k. Regardless of joint number that a spring crosses by, $\boldsymbol{K}^{ik}$ only possesses non-zero component matrices with six distinct values as shown in Fig. 3 labeled with different shades. To achieve static balancing, only off-diagonal stiffness component matrices are shown in solid shade, which has to be taken into account. Therefore, only four distinct non-zero component matrices are actually considered.

If a spring is installed between two adjacent links which spans over only one articulated joint, it is called a mono-articular spring. Mono-articular spring corresponds to three distinct non-zero component matrices, as well as Eqs.(7b) and (8a,c), in the elastic stiffness block matrix $\boldsymbol{K}^{ik}$. Only the one lies in the off-diagonal elastic stiffness block matrix is actually considered, as shown in Fig. 3(b). The number of mono-articular springs corresponds to the super-diagonal elements in the spring configuration matrix $\boldsymbol{\Lambda}$.

Under the assumption that the mass center of links is aligned on the line passing through the joints of links, the four distinct values of non-zero component matrices can be discussed in two separate parts, signs and magnitudes. The sign of the value of non-zero component matrices is determined by the attachment angles of springs α^{ik} and β^{ik}, and the difference in magnitude of the value of non-zero component matrices corresponds to the attachment points of the spring.

Referring to the effect of attachment angles, there are four kinds of distribution features of positive and negative signs in $\boldsymbol{K}^{ik}$, where $(\alpha^{ik}, \beta^{ik})$ equals to $(0°, 0°)$, $(0°, 180°)$, $(180°, 0°)$, and $(180°, 180°)$ as shown in Fig. 4. $\alpha^{ik} = 0°$ leads to negative signs which appear in the off-diagonal first non-zero row, while $\beta^{ik} = 180°$ results in negative signs, which contribute to the off-diagonal last non-zero column. For both $\alpha^{ik} = 0°$ and $\beta^{ik} = 180°$, the value of component of elastic stiffness block matrix is located in the first non-zero row, in the last non-zero column which is positive. For ground-attached spring, according to Eq.(26a,b), the attachment angle α^{1k} has a difference of 90° for contributing same signs of the values of non-zero terms in elastic stiffness block matrix.

Consider the distribution of magnitude of values of non-zero component matrices in $\boldsymbol{K}^{ik}$ as shown in Fig. 5. When both sign and magnitude of values in $\boldsymbol{K}^{ik}$ are discussed, there are three special cases of distribution feature. For a spring attached on the joint of link i, a^{ik} equals to r_i and $(\alpha^{ik}, \beta^{ik})$ equals to $(180°, 0°)$ or $(180°, 180°)$, the non-zero terms result in two among four distinct component matrices. Its distribution feature is shown in Fig. 6(a). For a spring installed on the joint of link k, b^{ik} equals to r_k and $(\alpha^{ik}, \beta^{ik})$ equals to $(0°, 0°)$ or $(180°, 0°)$. Similarly, the non-zero terms result in only two among four distinct component matrices as represented in Fig. 6(b); for one end of the spring attached on the joint of link i and the other installed on the joint of link k, a^{ik} equals to r_i, and b^{ik} equals to r_k at the same time, and (α^{ik},β^{ik}) equals to $(180°, 0°)$, contributing to only single type of off-diagonal non-zero terms, as shown in Fig. 6(c).

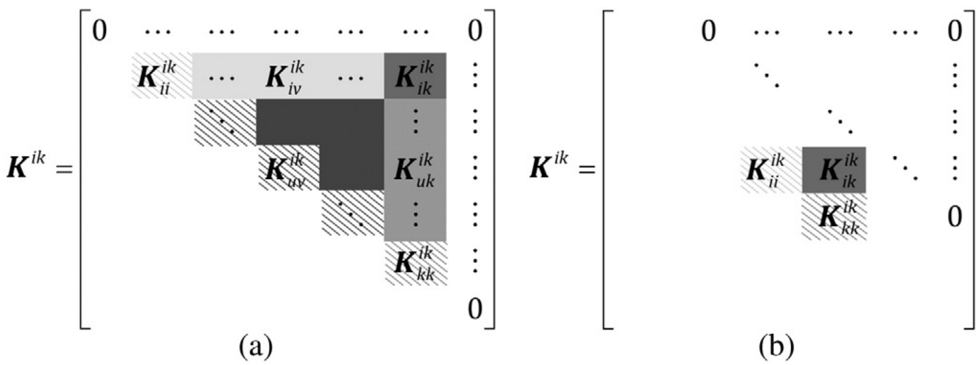


Fig. 3. Distribution features of non-zero component matrices in elastic stiffness block matrix according to the type of spring: (a) multi-articular spring; (b) mono-articular spring.

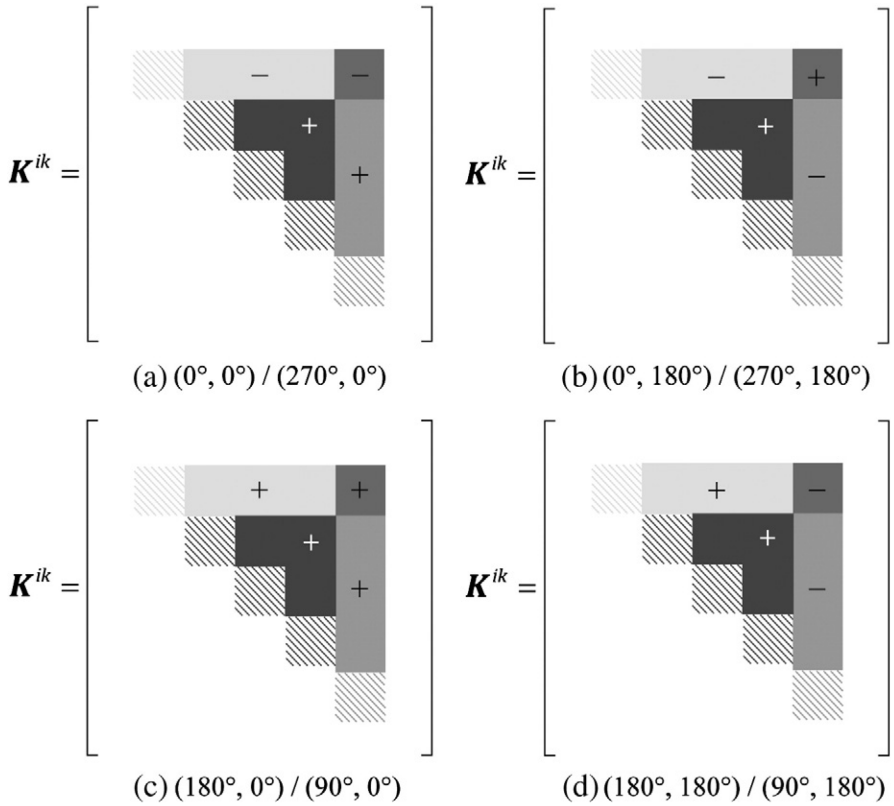


Fig. 4. Distribution features of signs of the values of non-zero terms in elastic stiffness block matrix according to the effect of attachment angles (α^{ik},β^{ik}).

5.2. Equivalent spring installation

For mono-articular springs, according to Eq.(7b), regardless of the installation angles α^{ik} and β^{ik}, the only off-diagonal component matrix $\mathbf{K}_{ik}^{ik}$ remains identical if $\beta^{ik} - \alpha^{ik}$ is constant. When this condition is satisfied, springs with different installation angles, as shown in Fig. 7, produce identical elastic potential effect. For instance, under the assumption that mass centers align along the line passing through the joints of links, installing a (0°, 0°) mono-articular spring is equivalent to embedding a (180°, 180°) mono-articular spring, and installing a (0°, 180°) one is equivalent to embedding a (180°, 0°) one; for ground-attached

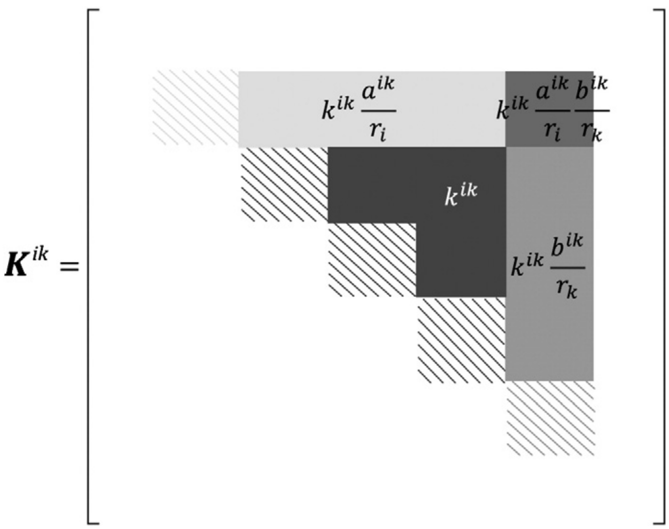


Fig. 5. Distribution of magnitude of values of non-zero terms in elastic stiffness block matrix.

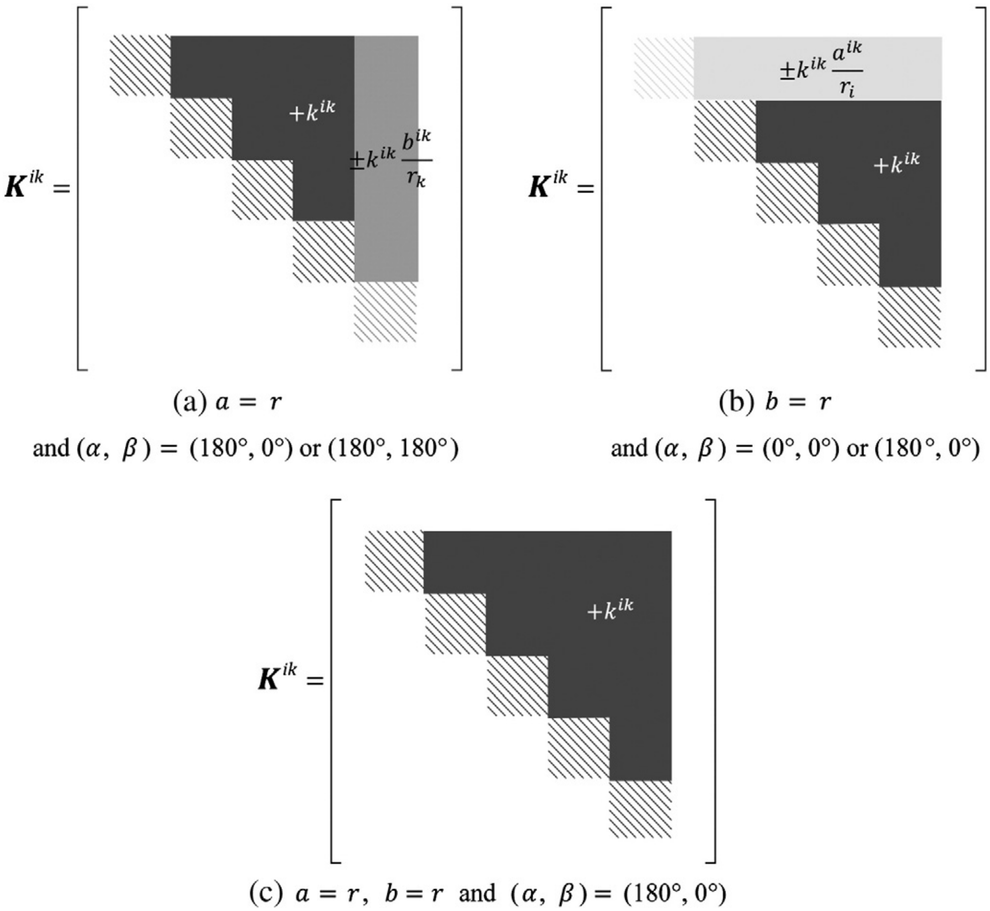


Fig. 6. Special cases of distribution features of non-zero terms in elastic stiffness block matrix with specific attachment points and attachment angles of springs.

springs, a mono-articular spring with (90°, 0°) is equivalent to that of with (270°, 180°), and a (90°, 180°) mono-articular spring is equivalent to a (270°, 0°) one.

Installing only one spring between two links is called single installation. Attaching more than one spring between two identical links is said to be multiple installation. Among which, embedding two springs between identical two links is double installation; installing three springs between same two links is triple installation, and so on. Based on former analyses, installing two springs between links i and k can be equivalent to attaching a spring between links i and k and another spring between links $i + 1$ and k as shown in Eq. (28a). This is because such two single installations can produce same elastic potential effect as a

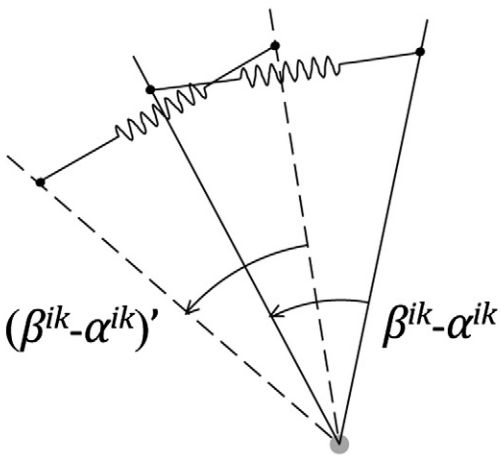


Fig. 7. Equivalent mono-articular springs.

double installation by possessing different spring constant, attachment points, or attachment angles. In Eq. (28a), the single installation corresponding to $\lambda_{(i+1)k} = 1$ is restricted to the springs with design parameters $a^{ik} = r_i$ and $(\alpha^{ik},\beta^{ik}) = (180°, 0°)$ or $(180°, 180°)$, as the special case shown in Fig. 6(a). There is a similar characteristic for installing a spring between links i and k and another between links i and $k-1$, as shown in Eq. (28b). The single installation corresponding to $\lambda_{i(k-1)} = 1$ is restricted to the springs with design parameters $b^{ik} = r_k$ and $(\alpha^{ik},\beta^{ik}) = (0°, 0°)$ or $(180°, 0°)$, the special case shown in Fig. 6(b). The equality which two single installations as shown in Eq. (28a) is equivalent to that of as shown in Eq. (28b) holds if and only if Eqs. (28a) and (28b) are associated with same non-zero terms in summation of elastic stiffness block matrices. For example, for three adjacent links, installing a mono-articular spring and a bi-articular spring on link a, as shown in Fig. 8(a), is equivalent to installing those on link c, as shown in Fig. 8(b).

$$= \begin{bmatrix} 0 & & & & \\ & \ddots & & 2 & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \Leftrightarrow = \begin{bmatrix} 0 & & & & \\ & \ddots & & 1 & \\ & & \ddots & 1 & \\ & & & \ddots & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}. \tag{28a}$$

$$= \begin{bmatrix} 0 & & & & \\ & \ddots & & 2 & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \Leftrightarrow = \begin{bmatrix} 0 & & & & \\ & \ddots & 1 & 1 & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}. \tag{28b}$$

6. General criteria of the admissible spring configuration

In this section, general criteria G1–G5 of the admissible spring configuration matrices for the design of an n-link statically balanced planar articulated manipulator are investigated. The order of determination of spring configuration is as follows: the first step deals with the top-right corner of the spring configuration matrix; secondly, the diagonal entries λ_{uv} where $u + v = 2n - 1$; then, the diagonal entries λ_{uv} where $u + v = 2n - 2$; the rest may be deduced by analogy. In addition, for entries on the same diagonal line, those lie in the first row or the last column are top priorities.

Consider the top-right corner of $\mathbf{K}$. The non-zero component matrix $\mathbf{K}^g_{1n}$ as shown in Eq. (20) has a negative value. In order to balance the off-diagonal terms for $\mathbf{K}$ to be zero, it is necessary to install at least one spring between ground and end link. In addition, this specific basic spring has to contribute a component matrix $\mathbf{K}^{ik}_{1n}$ with a positive value. Hence, according to Fig. 4, the attachment angles of the spring must be equal to $(90°, 0°)$ or $(270°, 180°)$. The general criterion G1 as shown in Table 1(a) is obtained.

The spring, which is required to fit between ground and end link, produces non-zero component matrices in all entries of stiffness block matrix. In order to compensate for the off-diagonal terms especially focusing on the last column of the spring configuration matrix, one more spring needs to be attached between ground and end link or between the ground-pivoted link and end link. If the last column of the stiffness block matrix remains non-zero, one more spring is required to be installed between the ground-pivoted link and end link or between the third link and end link. The rest may be deduced by analogy. The general criterion G2 as shown in Table 1(a) is derived.

According to the distribution feature of non-zero component matrices in elastic stiffness block matrix, as shown in Fig. 3, each spring produces non-zero component matrices with two distinct values at most in the first non-zero row of the off-diagonal elastic stiffness block matrix. However, the gravitational stiffness block matrix possesses $n - 1$ distinct component matrices in the first row. Hence, every two connected links must have at least one ground-attached spring to balance the first row of gravitational stiffness block matrix. The general criterion G3 is shown in Table 1(a).

According to G3, considering the distribution of positive and negative values of non-zero terms in $\mathbf{K}^{ik}$ associated with the attachment angles, the feasible ground-attached spring installation configuration corresponding to λ_{1v} is discussed. Since the non-zero terms contributed in the first row of $\mathbf{K}^{ik}$ by the spring, are expected to balance the non-zero terms with negative values in the first row of $\mathbf{K}^g$, the distribution of positive and negative values of non-zero terms in $\mathbf{K}^{ik}$ produced by the ground-attached

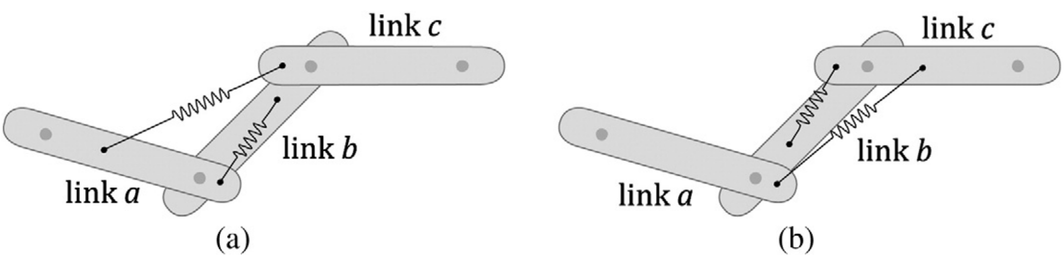


Fig. 8. Equivalent spring configuration installation for three adjacent links.

Table 1
Criteria of the admissible spring configuration matrices for the design of an n-link statically balanced planar articulated manipulator.

(a) General criteria	
G1.	λ_{1n} must not be less than 1. A basic spring with attachment angles (α^{1n},β^{1n}) equal to (90°, 0°) or (270°, 180°) is required to fit in between ground and end link.
G2.	At least two springs are required to be installed on the end link, two attached between ground and end link or one installed between ground and end link and another attached between the ground-pivoted link and end link. The elements in the last column of the spring configuration matrix have to satisfy $\sum_{j=1}^{n-1}\lambda_{jn} \geq 2$.
G3.	Every two adjacent links must have at least one ground-attached spring. That is, total number of ground-attached springs must be greater than $\lceil (n-1)/2 \rceil$, where $\lceil * \rceil$ denotes the ceiling function. Each element in the first row of the spring configuration matrix must satisfy $\lambda_{1v} + \lambda_{1(v+1)} \geq 1$ for $v = 2, \ldots, n-1$.
G4.	λ_{1v} do not correspond to ground-attached springs with attachment angles (270°, 0°). For entry $(1, v^*)$ where $3 \leq v^* \leq n$, if $\lambda_{1v^*} = 0$ or $\lambda_{1v^*} \neq 0$ corresponding to $\sum \mathbf{K}^{ik}_{1(v^*-1)}$ with negative value, $\lambda_{1(v^*-1)}$ must be at least equal to 1 corresponding to springs with attachment angles (90°, 0°) or (270°, 180°); if $\lambda_{1v^*} \neq 0$ corresponding to $\sum \mathbf{K}^{ik}_{1(v^*-1)}$ with positive value, $\lambda_{1(v^*-1)}$ can be equal to 0 or greater than 1 corresponding to springs with attachment angles (90°, 0°), (90°, 180°), or (270°, 180°).
G5.	For any entry (u^*, v^*) where $u^* \neq 1$, if $\sum_{u=1}^{u^*-1}\lambda_{uv^*} + \sum_{v=v^*+1}^{n}\lambda_{u^*v} = 0$ or λ_{uv^*} for $u \leq u^* - 1$ do not correspond to any spring with attachment angle β^{uv^*} equal to 180°, and λ_{u^*v} for $v \geq v^* + 1$ do not correspond to any spring with attachment angle α^{u^*v} equal to 0°, $\lambda_{u^*v^*}$ cannot be zero. $\lambda_{u^*v^*}$ has to be at least 1 associated with a spring with $\left(\alpha^{u^*v^*},\beta^{u^*v^*}\right)$ equal to (0°, 0°) or (180°, 180°).
(b) Additional criteria (depending on specific design objectives)	
Design objective: minimum number of springs	
S1.	λ_{1n} must not be less than 1. A spring with attachment angles (α^{1n},β^{1n}) equal to (90°, 0°) is required to be fitted between ground and end link.
S2.	$\sum(\lambda_{1n} + \lambda_{2n})$ must be equal to 2. A spring with attachment angles (α^{2n},β^{2n}) equal to (180°, 180°) and a^{2n} equal to r_2 fitted between ground-pivoted and end link, or that with attachment angles (α^{1n},β^{1n}) equal to (90°, 180°) or (270°, 180°) installed between ground and end link is required.
S3.	λ_{1v} has to be equal to 1, for $v = 3, \ldots, n-1$. Only ground-pivoted link and end link do not need to be installed a ground-attached spring with attachment angles (α^{1v},β^{1v}) equal to (90°, 180°).
Design objective: minimum total number of articulated joints which springs span over	
J1.	Only single spring can be installed between a pair of distinct links. λ_{uv} cannot be greater than 1. $\lambda_{uv} \leq 1$ for $u, v = 1, \ldots, n$.
J2.	A spring with attachment angles (α^{1n},β^{1n}) equal to (90°, 0°) is required to be fitted between ground and end link. λ_{1n} must be equal to 1.
J3.	A spring is required to be fitted between ground-pivoted and end link. λ_{2n} must equal to 1. The attachment angles of the spring (α^{2n},β^{2n}) equal to (180°, 180°), and a^{2n} equals to r_2.
J4.	If λ_{1v^*} does not correspond to any springs with attachment angle β^{1v^*} equal to 180°, entry lies in the second row, column v^* must be equal to 1, corresponding to a spring with attachment angles (α^{2v},β^{2v}) equal to (180°, 180°) and a^{2v} equal to r_2.

spring with attachment angles (270°, 0°) as shown in Fig. 4(a) is not appreciated. Particularly when compared with that of produced by the ground-attached spring with attachment angles (90°, 0°) as represented in Fig. 4(c). In addition, the magnitude of non-zero terms in the first row of $\mathbf{K}^g$ are in an ascending order from $\mathbf{K}^g_{1n}$ to $\mathbf{K}^g_{12}$. In order to zero those non-zero terms, $\sum \mathbf{K}^{ik}$ must possesses non-zero terms with positive values in the first row and with magnitude of an ascending order from $\sum \mathbf{K}^{ik}_{1n}$ to $\sum \mathbf{K}^{ik}_{12}$. If none of the ground-attached springs is installed on link v^* or if some are but correspond to $\sum \mathbf{K}^{ik}_{1v^*}$ with negative value, it has to install at least one ground-attached spring with attachment angles (90°, 0°) or (270°, 180°) on link $v^* - 1$ to make $\sum \mathbf{K}^{ik}_{1(v^*-1)}$ with positive value greater than $\sum \mathbf{K}^{ik}_{1v^*}$. If there is more than or equal to one ground-attached spring installed on link v^* already corresponds to $\sum \mathbf{K}^{ik}_{1v^*}$ with positive value, none of ground-attached springs installed on link $v^* - 1$ is allowable. The general criterion G4 is concluded as shown in Table 1(a).

The component matrix $\mathbf{K}_{u^*v^*}$ beyond the first row and the last column is associated with the springs attached on link u^* or link v^*, and the springs that span over links u^* and v^*. Consider the springs that are not attached to the ground. For springs spanning over links u^* and v^*, according to Eqs.(26c) and (27c), springs installed between link u that precedes link u^* and link v that is beyond link v^* contribute non-zero terms with positive sign to the value of $\mathbf{K}_{u^*v^*}$ as shown in Fig. 4. Hence, in order to zero each component matrix in off-diagonal stiffness block matrix out of the first row, springs that contribute negative sign to the value of $\mathbf{K}_{u^*v^*}$ are required. According to Eqs.(26b) and (27b), springs installed between links u^* and v^* associated with $\left(\alpha^{u^*v^*},\beta^{u^*v^*}\right)$ equal to (0°, 0°) or (180°, 180°) as shown in Fig. 4(a) and (d) contribute negative sign to the value of $\mathbf{K}_{u^*v^*}$. From Eqs.(26d) and (27d), springs attached between link u that precedes link u^* and link v^* associated with β^{uv^*} and equal to 180°, as shown in Fig. 4(b) and (d), which also contribute negative sign to the value of $\mathbf{K}_{u^*v^*}$. According to Eq.(27a), the springs attached between link u^* and link v that is beyond link v^* and associated with α^{u^*v} equal to 0° as shown in Fig. 4(a) and (b). As a result, a (0°, 0°) spring or a (180°, 180°) spring between links u^* and v^* have to be installed unless there is a spring attached from link preceding link u^* to link v^* or from link u^* to link succeeding link v^* that contributes negative sign to the value of $\mathbf{K}_{u^*v^*}$. The general criterion G5 is obtained, as shown in Table 1(a).

7. Specific design objectives for spring installation

7.1. Minimum number of springs

The admissible spring configuration with minimum number of springs can be further determined. To achieve static balancing using minimum number of springs, each spring must be utilized to maintain balance. Based on this principle, three additional criteria S1–S3 are obtained.

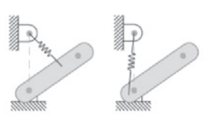
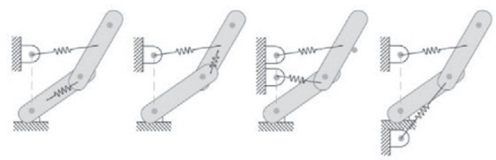
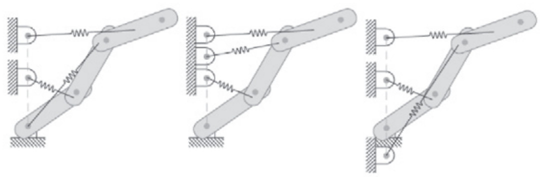
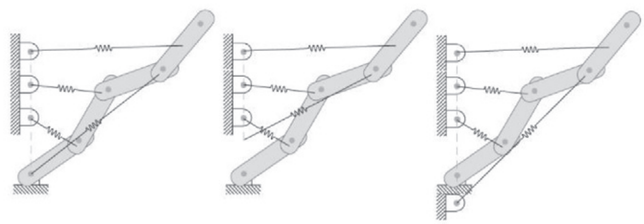
According to G1, at least one spring has to be installed between ground and end link in order to maximize the contribution of elastic potential effect to achieve static balancing. The attachment angles (α^{1n},β^{1n}) have to be equal to (90°, 0°), since it produces non-zero terms with positive values in the first row of stiffness block matrix as shown in Fig. 4(c). The additional criterion S1 as shown in Table 1(b) is derived.

As the spring embedded between ground and end link produces non-zero terms in all component of stiffness block matrix, another spring is required to compensate the non-zero terms produced in the second row, last column. According to G2, at least two springs are necessary to be attached on end link. Except for the specific spring required based on S1, one more spring has to be installed between ground and end link or between ground-pivoted link and end link. In order to maximize the elastic potential effect to achieve static balancing, the spring has to produce $n - 1$ terms with negative sign and same magnitude of the value to zero the non-zero terms with positive sign in the last column. Hence, according to Fig. 4(b) and (d), the required spring attached between ground and end link is associated with β^{1n} equal to 180°. From Fig. 6(a), the required spring installed between ground-pivoted link and end link is associated with (α^{2n},β^{2n}) equal to (180°, 180°), and a^{2n} equals to r_2. The additional criterion S2 is shown in Table 1(b).

According to G3, every two adjacent links must have at least one ground-attached spring. In addition, in order to zero the rest of non-zero component matrices in **K** using minimum number of springs, from G4 and G5, it can be found that installing ground-attached springs associated with attachment angles equal to (90°, 180°) as shown in Fig. 4(d) on each links except for ground-pivoted link and end link gains maximum profit. The additional criterion S3 as shown in Table 1(b) is obtained.

Based on the general criteria and three additional criteria above, the admissible configuration matrices for 1-, 2-, 3-, and 4-DOF statically balanced articulated manipulator are tabulated in Table 2 with corresponding schematic diagrams. From S1 and S2, each n-link statically balanced articulated manipulator corresponds to two admissible configuration matrices. One with both the first and second row of the last column equal to 1, where $\lambda_{1n} = 1$ corresponds to a (90°, 0°) spring and $\lambda_{2n} = 1$ corresponds to a (180°, 180°) spring. The other admissible configuration matrix corresponds to double installation between ground and end link with a (90°, 0°) spring and a (90°, 180°) or (270°, 180°) spring. For n equal or greater than 4, according to S3, the rest of the components of configuration matrix can then be determined.

Table 2
Admissible configuration matrices with minimum number of spring and the corresponding schematic diagrams.

n=2 (1-DOF)		n=3 (2-DOF)	
$\begin{bmatrix} 0 & 1^* \\ & 0 \end{bmatrix}$		$\begin{bmatrix} 0 & 0 & 1 \\ & 0 & 1^* \\ & & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2^* \\ & 0 & 0 \\ & & 0 \end{bmatrix}$	
n=4 (3-DOF)			
$\begin{bmatrix} 0 & 0 & 1 & 1 \\ & 0 & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 2^* \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}$			
n=5 (4-DOF)			
$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ & 0 & 0 & 0 & 1 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 & 2^* \\ & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix}$			

* denotes the spring has two ways of installation with different attachment angles.

Considering the equivalent spring installation for mono-articular spring, (90°, 0°) spring required for 2-links manipulators can be replaced by a (270°, 180°) spring. Similarly, the mono-articular spring with attachment angles (180°, 180°) required for the 3-links manipulators can be substituted for that with attachment angles (0°, 0°).

The minimum numbers of required spring are 1, 2, 3, and 4 for 1-, 2-, 3-, and 4-DOF statically balanced articulated manipulator respectively. The minimum number of springs required for an n-link system is shown as:

$$\min N = n-1. \tag{29}$$

The results of admissible spring configuration matrices are different from that of proposed by Lin et al. [17]. Lin et al. suggested k is specified to be predetermined so that only four parameters are considered in design equations. In addition, only single installation is available. The minimum number of springs suggested is equal to $\lceil n(n-1)/4 \rceil$. In contrast, the proposed admissible spring configuration matrices in this paper are based on two assumptions. Firstly, k^{ik}, a^{ik}, b^{ik}, α^{ik}, β^{ik} are five parameters of design equations. Within, α^{ik} and β^{ik} only have two choices. Secondly, multiple installation is available, which means that elements in the spring configuration matrix $\mathbf{\Lambda}$ are not restricted to 0 or 1.

From another aspect, the minimum number of springs with direct spring installation proposed in this paper is equal to that of the auxiliary link method as suggested by Streit and Shin [20]. From a practical point of view, installing equal number of springs to achieve static balancing with direct spring installation is appreciated as without adding $2(n-2)$ auxiliary links into the system. However, if the direct spring installation method is used, since each link except for ground-pivoted link must be installed at least one ground-attached spring, the number of springs that span over great distance may increase while the degrees of freedom of the statically balanced articulated manipulators become higher.

7.2. Minimum total number of articulated joints which springs span over

As springs with larger spring constant are harder to be elongated, as large number of articulated joints in which a spring spans over is not appropriate. In addition, the fewer number of total articulated joints a spring spans over, the less likely of interference. Hence, we aim to consider another design objective, which is to minimize the total number of articulated joints that springs span over. According to the analyses of gravitational effect of manipulator and elastic potential effect of spring, the criteria of admissible spring configuration can be obtained. The design objective function is shown as:

$$\min \sum\nolimits_N (k-i). \tag{30}$$

According to the characteristic of equivalent spring installation for multiple installations of springs, as shown in Eqs. (28a) and (28b), a multiple installation can be replaced by two single installations. In order to minimize the total number of articulated joints which springs span over, only single installation is available. The additional criterion J1 as shown in Table 1(b) is obtained.

From G1 and J1 above, one spring is required to be installed between ground and end link. In order to gain the largest elastic potential effect to balance the non-zero terms with negative values in the first row of gravitational stiffness block matrix, the attachment angles (α^{1n},β^{1n}) of the spring have to be equal to (90°, 0°). The additional criterion J2 is shown in Table 1(b).

According to J2, it is necessary to attach a (90°, 0°) spring between ground and end link. Since the spring contributes non-zero terms with positive values in each entries of the stiffness block matrix as shown in Fig. 4(c), another spring is required to compensate the non-zero terms that appear in the last column. With target of minimum total number of articulated joints, which springs span over, a (180°, 180°) spring attached between the ground-pivoted link and end link is required. In addition, to balance the non-zero terms in the last column from the second row to the row $n-1$ at the same time, the spring has to be attached on the joint of ground-pivoted link according to Fig. 6(a), a^{2n} equals to r_2. The additional criterion J3 as shown in Table 1(b) is derived.

Based on J3, considering the entries in the second row of spring configuration matrix according to G5, there is a spring corresponding to $\lambda_{2n}=1$ but without attachment angle α^{2n} equal to 0°. Hence, for entry (2, v^*), where λ_{1v^*} does not corresponding to any spring with attachment angle β^{1v^*} equal to 180°, λ_{2v^*} must be equal to 1, corresponding to a (180°, 180°) spring that produces non-zero terms with negative values and with same magnitude in the last non-zero column of stiffness block matrix, as shown in Fig. 6(a). The additional criterion J4 as shown in Table 1(b) is obtained.

From G3, G4 and J4 with the design objective of minimum total number of articulated joints which springs span over, the feasible ground-attached spring installation configuration associated with specific attachment angles can be generated. For $\lambda_{1v^*}=1$ corresponding to a (90°, 0°) or (90°, 180°) spring,

$$\lambda_{1(v^*-1)} = \begin{cases} 0 \textit{ or } 1 \textit{ with a } (90^\circ, 180^\circ) \textit{ spring} & v^* = 5, \ldots, n \\ 0 \textit{ or } 1 \textit{ with a } (90^\circ, 180^\circ) \textit{ or } (90^\circ, 0^\circ) \textit{ spring} & v^* = 4 \\ 0 & v^* = 3 \end{cases}. \tag{31}$$

For $\lambda_{1v^*} = 0$ or $\lambda_{1v^*} = 1$ corresponding to a (270°, 180°) spring,

$$\lambda_{1(v^*-1)} = \begin{cases} 1 \textit{ with a } (270^\circ, 180^\circ) \textit{ spring} & v^* = 5, \ldots, n \\ 1 \textit{ with a } (270^\circ, 180^\circ) \textit{ or } (90^\circ, 0^\circ) \textit{ spring} & v^* = 3, 4 \end{cases}. \tag{32}$$

According to the general criteria and the additional criteria above, the admissible configuration matrices for 1-, 2-, 3-, and 4-DOF statically balanced articulated manipulator with minimum total number of articulated joints which springs span over are tabulated in Table 3. The minimum total number of articulated joints which springs span over can be expressed as

$$\min \sum\nolimits_N (k-i) = \frac{n^2+n-6}{2} \quad \text{for} \quad n \geq 3 \tag{33}$$

Based on J2 and J3, for each n-link statically balanced articulated manipulator, both the first and second rows of the last column in admissible configuration matrices are equal to 1, where 1 which lies in λ_{1n} corresponds to a (90°, 0°) spring, and another which lies in λ_{2n} corresponds to a (180°, 180°) spring with a^{2n} equals to r_2. From J4 and Eqs. (31) and (32), the rest of the elements of configuration matrices for $n = 4$ and 5 can be obtained.

In addition, for $n = 4$ and 5, considers the specific admissible configuration matrix with both λ_{23} and λ_{24} equal to 1. Due to the characteristic of equivalent spring installation, the corresponding installation of a mono-articular spring and a bi-articular spring attached on link 2 can be replaced by that of attached on link 4, as shown in Fig. 8. Hence, it results an additional admissible configuration matrix for both $n = 4$ and 5, corresponding to a (0°, 0°) bi-articular spring attached between links 2 and 4 and a (0°, 0°) or a (180°, 180°) mono-articular spring installed between links 3 and 4. Considering the equivalent spring installation of mono-articular springs, the only (90°, 0°) spring required for 2-links manipulators can be substituted for a (270°, 180°) one. Similarly, every mono-articular spring with attachment angles (180°, 180°) can be replaced by that of with attachment angles (0°, 0°).

Note that there is one admissible configuration matrix for each n-link statically balanced articulated manipulator which, at the same time, fits both objectives of minimum number of springs and minimum number of articulated joints which springs span over. The spring configuration that installs one ground-attached spring on each link except for ground-pivoted link and additionally attaches one spring between ground-pivoted link and end link is the optimal choice taking into account the two design objectives.

8. Illustrative example of a 3-DOF planar articulated manipulator

A 3-DOF planar articulated manipulator that intends to be statically balanced is considered as an illustrative example. The inertia and dimensional parameters are tabulated in Table 4 and the associated schematic plot is presented in Fig. 9(a). The non-zero component matrices in the off-diagonal upper triangular gravitational stiffness matrix are as follows:

$$\mathbf{K}^g_{12} = -\left(\frac{s_2}{r_2} m_2 + m_3 + m_4\right) g\mathbf{I}' \tag{34a}$$

$$\mathbf{K}^g_{13} = -\left(\frac{s_3}{r_3} m_3 + m_4\right) g\mathbf{I}' \tag{34b}$$

$$\mathbf{K}^g_{14} = -\left(\frac{s_4}{r_4} m_4\right) g\mathbf{I}'. \tag{34c}$$

Table 3
Admissible configuration matrices with minimum total number of articulated joints which springs span over.

$n = 2$ (1-DOF)	$n = 3$ (2-DOF)	$n = 4$ (3-DOF)	$n = 5$ (4-DOF)
$\begin{bmatrix} 0 & 1^* \\ & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ & 0 & 1^* \\ & & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1^* & 1 \\ & 0 & 0 & 1^* \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1^* & 0 & 1 \\ & 0 & 1^* & 1 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1^* & 0 & 1 \\ & 0 & 0 & 1 \\ & & 0 & 1^* \\ & & & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 1^* & 1 \\ & 0 & 0 & 0 & 1^* \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1^* & 0 & 1 & 1 \\ & 0 & 1^* & 0 & 1 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1^* & 1 & 0 & 1 \\ & 0 & 0 & 1 & 1 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ & 0 & 1^* & 1 & 1 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ & 0 & 0 & 1 & 1 \\ & & 0 & 1^* & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix}$

* denotes the spring has two ways of installation with different attachment angles.

Table 4
Inertia and dimensional parameters of the 3-DOF manipulator.

j	m_j (kg)	r_j (m)	s_j (m)
2	0.848	0.180	0.090
3	0.848	0.180	0.090
4	1.330	0.070	0.060

For design objective that minimizes the number of springs, the spring configuration matrix, based on the result of Table 2, can be shown as:

$$\Lambda = \begin{bmatrix} 0 & 0 & 1 & 2 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}. \tag{35}$$

Its associated spring attachment angles are listed in Table 5(a). The non-zero component matrices in the off-diagonal upper triangular elastic stiffness matrix contributed by three springs respectively can be derived accordingly.

$$\mathbf{K}^{13}\begin{cases} \mathbf{K}_{12}^{13} = k^{13}a^{13}\ \mathbf{I}' \\ \mathbf{K}_{13}^{13} = -k^{13}a^{13}\dfrac{b^{13}}{r_3}\ \mathbf{I}' \\ \mathbf{K}_{23}^{13} = -k^{13}\dfrac{b^{13}}{r_3}\mathbf{I} \end{cases}. \tag{36}$$

$$\mathbf{K}^{14_1}\begin{cases} \mathbf{K}_{12}^{14_1} = \mathbf{K}_{13}^{14_1} = k^{14_1}a^{14_1}\ \mathbf{I}' \\ \mathbf{K}_{14}^{14_1} = k^{14_1}a^{14_1}\dfrac{b^{14_1}}{r_4}\ \mathbf{I}' \\ \mathbf{K}_{23}^{14_1} = k^{14_1}\mathbf{I} \\ \mathbf{K}_{24}^{14_1} = \mathbf{K}_{34}^{14_1} = k^{14_1}\dfrac{b^{14_1}}{r_4}\mathbf{I} \end{cases}. \tag{37}$$

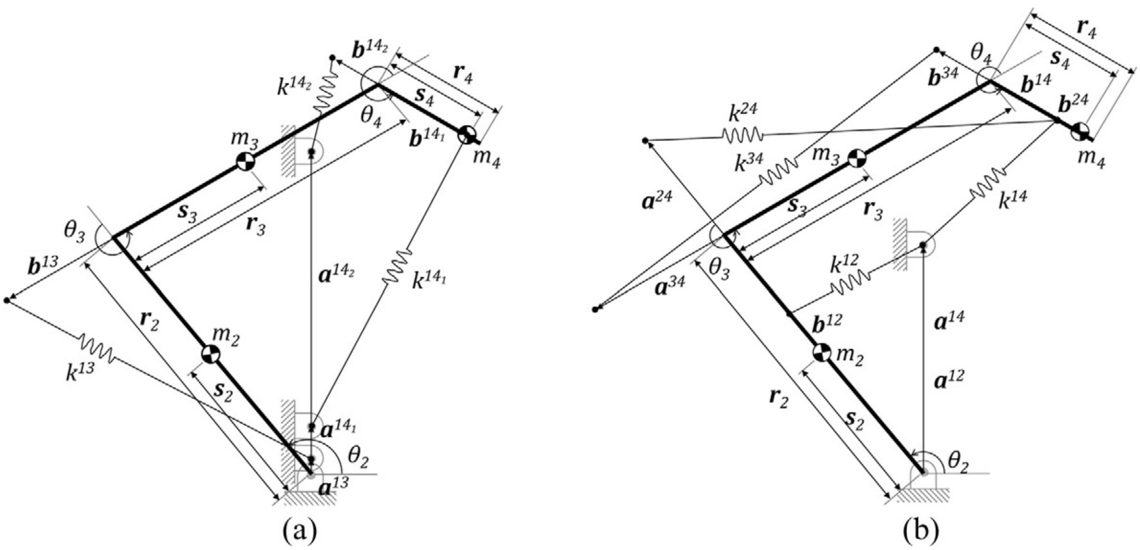


Fig. 9. 3-DOF statically balanced manipulator with specific design objectives: (a) minimum number of springs; (b) minimum total number of articulated joints which springs span over.

Table 5
Spring design parameters of the 3-DOF manipulator for specific design objectives.

ik	k^{ik} (N/m)	a^{ik} (m)	b^{ik} (m)	α^{ik} (°)	β^{ik} (°)
(a) Minimum number of springs					
13	590	0.010	0.073	90	180
14_1	78	0.190	0.063	90	0
14_2	160	0.030	0.031	90	180
(b) Minimum total number of articulated joints which springs span over					
12	93	0.134	0.120	90	0
14	128	0.134	0.045	90	0
24	320	0.072	0.045	0	0
34	400	0.262	0.035	180	180

$$\mathbf{K}^{14_2}\begin{cases}\mathbf{K}_{12}^{14_2}=\mathbf{K}_{13}^{14_2}=k^{14_2}a^{14_2}\ \mathbf{I}' \\ \mathbf{K}_{14}^{14_2}=-k^{14_2}a^{14_2}\dfrac{b^{14_2}}{r_4}\ \mathbf{I}' \\ \mathbf{K}_{23}^{14_2}=k^{14_2}\mathbf{I} \\ \mathbf{K}_{24}^{14_2}=\mathbf{K}_{34}^{14_2}=-k^{14_2}\dfrac{b^{14_2}}{r_4}\mathbf{I}\end{cases}. \tag{38}$$

According to Eqs. (23), (34a), (34b), (34c) and (36)-(38), to zero the upper triangular stiffness block matrix, the design equations can be obtained as:

$$\mathbf{K}_{12}=\left[-\left(\frac{s_2}{r_2}m_2+m_3+m_4\right)g+k^{13}a^{13}+k^{14_1}a^{14_1}+k^{14_2}a^{14_2}\right]\ \mathbf{I}'=\mathbf{0} \tag{39}$$

$$\mathbf{K}_{13}=\left[-\left(\frac{s_3}{r_3}m_3+m_4\right)g-k^{13}a^{13}\frac{b^{13}}{r_3}+k^{14_1}a^{14_1}+k^{14_2}a^{14_2}\right]\ \mathbf{I}'=\mathbf{0} \tag{40}$$

$$\mathbf{K}_{14}=\left[-\left(\frac{s_4}{r_4}m_4\right)g+k^{14_1}a^{14_1}\frac{b^{14_1}}{r_4}-k^{14_2}a^{14_2}\frac{b^{14_2}}{r_4}\right]\ \mathbf{I}'=\mathbf{0} \tag{41}$$

$$\mathbf{K}_{23}=\left[-k^{13}\frac{b^{13}}{r_3}+k^{14_1}+k^{14_2}\right]\mathbf{I}=\mathbf{0} \tag{42}$$

$$\mathbf{K}_{24}=\mathbf{K}_{34}=\left[k^{14_1}\frac{b^{14_1}}{r_4}-k^{14_2}\frac{b^{14_2}}{r_4}\right]\mathbf{I}=\mathbf{0}. \tag{43}$$

By arranging Eqs. (39)-(43), the attachment points vary depending on spring constants. The spring constants correspond to a maximum elongation, which limits the specified workspace of the manipulator, should be considered. The spring constants k^{13}, k^{14_1}, and k^{14_2} are determined through trial-and-error with appropriate installation parameters a^{13}, b^{13}, a^{14_1}, b^{14_1}, a^{14_2}, and b^{14_2}. The design parameters of springs are listed in Table 5(a).

This 3-DOF statically balanced planar articulated manipulator is simulated where the elastic potential energy function, and the gravitational potential energy functions are plotted in Fig. 10(a). The motion is initially with $\theta_2 = 135°$, $\theta_3 = 225°$, and $\theta_4 = 270°$. Next, the ground-pivoted link rotates vertically while links 3 and 4 remain horizontal and vertical respectively, where $\theta_2 = 90°$, $\theta_3 = 270°$, and $\theta_4 = 270°$. Finally, links 3 and 4 are raised to complete the motion, where $\theta_2 = 90°$, $\theta_3 = 315°$, and $\theta_4 = 45°$. The elastic potential energy and the gravitational potential energy are entirely transferred, thus the total potential energy remains constant equal to 20.56 N m throughout the motion.

For another design objective that minimize total number of articulated joints which springs span over, according to the result of Table 3, the spring configuration matrix can be shown as:

$$\Lambda=\begin{bmatrix}0 & 1 & 0 & 1\\ & 0 & 0 & 1\\ & & 0 & 1\\ & & & 0\end{bmatrix} \tag{44}$$

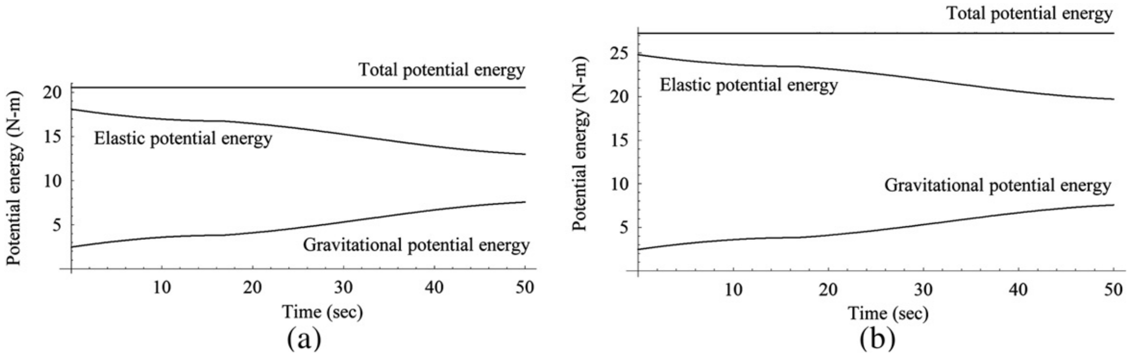


Fig. 10. Simulated potential energy through the motion: (a) minimum number of springs; (b) minimum total number of articulated joints which springs span over.

which is associated with spring attachment angles tabulated in Table 5(b). The schematic plot is shown as Fig. 9(b). The non-zero component matrices in the off-diagonal upper triangular elastic stiffness matrix generated by four springs respectively are obtained.

$$\mathbf{K}^{12}\left\{ \quad \mathbf{K}_{12}^{12} = k^{12}a^{12}\frac{b^{12}}{r_2}\ \mathbf{I}'. \right. \tag{45}$$

$$\mathbf{K}^{14}\left\{ \begin{array}{l} \mathbf{K}_{12}^{14} = \mathbf{K}_{13}^{14} = k^{14}a^{14}\ \mathbf{I}' \\ \mathbf{K}_{14}^{14} = k^{14}a^{14}\dfrac{b^{14}}{r_4}\ \mathbf{I}' \\ \mathbf{K}_{23}^{14} = k^{14}\mathbf{I} \\ \mathbf{K}_{24}^{14} = \mathbf{K}_{34}^{14} = k^{14}\dfrac{b^{14}}{r_4}\mathbf{I} \end{array} \right. . \tag{46}$$

$$\mathbf{K}^{24}\left\{ \begin{array}{l} \mathbf{K}_{23}^{24} = -k^{24}\dfrac{a^{24}}{r_2}\ \mathbf{I} \\ \mathbf{K}_{24}^{24} = -k^{24}\dfrac{a^{24}}{r_2}\dfrac{b^{24}}{r_4}\ \mathbf{I} \\ \mathbf{K}_{34}^{24} = k^{24}\dfrac{b^{24}}{r_4}\mathbf{I} \end{array} \right. . \tag{47}$$

$$\mathbf{K}^{34}\left\{ \quad \mathbf{K}_{34}^{34} = -k^{34}\frac{a^{34}}{r_3}\frac{b^{34}}{r_4}\ \mathbf{I}. \right. \tag{48}$$

To achieve static balancing, according to Eqs. (23), (34a), (34b), (34c) and (45)-(48), the design equations are as follows:

$$\mathbf{K}_{12} = \left[-\left(\frac{s_2}{r_2}m_2 + m_3 + m_4\right)g + k^{12}a^{12}\frac{b^{12}}{r_2} + k^{14}a^{14} \right]\ \mathbf{I}' = \mathbf{0} \tag{49}$$

$$\mathbf{K}_{13} = \left[-\left(\frac{s_3}{r_3}m_3 + m_4\right)g + k^{14}a^{14} \right]\ \mathbf{I}' = \mathbf{0} \tag{50}$$

$$\mathbf{K}_{14} = \left[-\left(\frac{s_4}{r_4}m_4\right)g + k^{14}a^{14}\frac{b^{14}}{r_4} \right]\ \mathbf{I}' = \mathbf{0} \tag{51}$$

$$\mathbf{K}_{23} = \left[k^{14} - k^{24}\frac{a^{24}}{r_2} \right]\mathbf{I} = \mathbf{0} \tag{52}$$

$$\mathbf{K}_{24} = \left[k^{14}\frac{b^{14}}{r_4} - k^{24}\frac{a^{24}}{r_2}\frac{b^{24}}{r_4} \right]\mathbf{I} = \mathbf{0} \tag{53}$$

$$\mathbf{K}_{34} = \left[k^{14}\frac{b^{14}}{r_4} + k^{24}\frac{b^{24}}{r_4} - k^{34}\frac{a^{34}}{r_3}\frac{b^{34}}{r_4}\right]\mathbf{I} = \mathbf{0}. \tag{54}$$

The suitable spring constants and installation parameters are listed in Table 5(b). Similarly, the variation of elastic and gravitational potential energy with same simulated motion is plotted in Fig. 10(b). The total potential energy equals to a constant 27.27 N m in any configuration.

9. Conclusion

This paper presents a design methodology to determine the spring installation configuration on a statically balanced planar articulated manipulator. The amended formulation of stiffness block matrices is proposed to deal with balance between gravitational potential energy and elastic potential energy. This paper discusses the elastic potential effect of spring, which is affected by the design parameters from two separate parts. The attachment angles of spring installation have an impact on the distribution of positive and negative value of non-zero terms in the elastic stiffness block matrix, while the attachment points of spring installation influence the distribution of magnitude of the value of non-zero terms in the elastic stiffness block matrix. The characteristic of equivalent spring installations, which produce same elastic potential effect, is also disclosed. Based on the analysis, general criteria of admissible spring configuration for n-link statically balanced articulated manipulator are derived. Furthermore, for different specific design objective, such as minimum number of springs and minimum total number of articulated joints which springs span over, the additional design criteria and admissible configuration matrices with 1-, 2-, 3-, and 4-DOF statically balanced planar articulated manipulators are obtained accordingly. The design of a 3-DOF statically balanced planar articulated manipulator is given as an illustrative design example.

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